

## SOME PROPERTIES OF A CLASS OF UNIVALENT FUNCTIONS DEFINED BY SUBORDINATION PROPERTY II

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ABSTRACT. In this paper, we study a class of univalent functions defined by subordination property. We obtain coefficient inequality, we also introduce the subclass  $H_{1,c_m}(A, B, \beta, q, s)$  consisting of functions with negative and fixed finitely many coefficients. We discuss some interesting properties of the class  $H_{1,c_m}(A, B, \beta, q, s)$ .

### 1. INTRODUCTION

Let  $J(n)$  denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0; n \in \mathbf{N}) \quad (1.1)$$

which are analytic and univalent in the open unit disk  $U = \{z \in \mathbf{C} : |z| < 1\}$ . If  $f \in J(n)$  is given by (1.1) and  $g \in J(n)$  is given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k, \quad (b_k \geq 0; n \in \mathbf{N}) \quad (1.2)$$

the convolution (or Hadamard product)  $(f * g)(z)$  of  $f$  and  $g$  is defined by

$$(f * g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k = (g * f)(z). \quad (1.3)$$

A function  $f \in J(n)$  is said to be univalently starlike of order  $\alpha$ , ( $0 \leq \alpha < 1$ ) in  $U$  if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha. \quad (1.4)$$

Similarly, a function  $f$  is univalently convex of order  $\alpha$ , ( $0 \leq \alpha < 1$ ) in  $U$  if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha. \quad (1.5)$$

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It is observed that  $f$  is convex function if and only if  $zf'$  is starlike function [2].

A function  $f \in J(n)$  is close-to-convex of order  $\alpha$  if

$$\operatorname{Re}\{f'(z)\} > \alpha, \quad (0 \leq \alpha < 1). \quad (1.6)$$

**Definition 1.1.** Let  $f$  and  $g$  be analytic in  $U$ . Then  $g$  is said to be subordinate to  $f$ , written  $g \prec f$  or  $g(z) \prec f(z)$ , if there exists a Schwarz function  $w$ , which is analytic in  $U$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in U$ ), such that  $g(z) = f(w(z))$  ( $z \in U$ ). Indeed it is known that

$$g(z) \prec f(z) \quad (z \in U) \Rightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$$

In particular, if the function  $f$  is univalent in  $U$ , we have the following equivalence ([6], [7]):

$$g(z) \prec f(z) \quad (z \in U) \Leftrightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$$

The operator

$$\begin{aligned} (H_s^q[a_1]f)(z) &= H_s^q(a_1, \dots, a_q; b_1, \dots, b_s)f(z) \\ &= {}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z) * f(z) \\ &= z + \sum_{k=n+1}^{\infty} \frac{(a_1)_{k-1} \cdots (a_q)_{k-1} a_k}{(b_1)_{k-1} \cdots (b_s)_{k-1} (k-1)!} z^k \\ &= z + \sum_{k=n+1}^{\infty} h(k) a_k z^k, \end{aligned} \quad (1.7)$$

where

$$h(k) = \frac{(a_1)_{k-1} \cdots (a_q)_{k-1}}{(b_1)_{k-1} \cdots (b_s)_{k-1} (k-1)!}. \quad (1.8)$$

Here  ${}_qF_s(z)$  is the generalized hypergeometric function for  $a_j \in \mathbf{C}$  ( $j = 1, 2, \dots, q$ ) and  $b_j \in \mathbf{C}$  ( $j = 1, 2, \dots, s$ ) such that  $b_j \neq 0, -1, -2, \dots$  ( $j = 1, 2, \dots, s$ ) defined by

$$\begin{aligned} {}_qF_s(z) &= {}_qF_s(a_1, \dots, a_q; b_1, \dots, b_s; z) \\ &= \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_q)_k}{(b_1)_k \cdots (b_s)_k k!} z^k, \quad (q \leq s+1, q, s \in \mathbf{N}_0, z \in U), \end{aligned} \quad (1.9)$$

where

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & (n \in \mathbf{N}). \end{cases}$$

The series  ${}_qF_s(z)$  in (1.9) converges absolutely for  $|z| < \infty$  if  $q < s+1$  and for  $|z| = 1$  if  $q = s+1$ . The linear operator defined in (1.7) is the Dziok - Srivastava operator (see [3], [4]) which contains the well-known operators like the Hohlov linear operator [6], the Carlson - Shafer operator [1], the Ruscheweyh derivative operator [12], the Srivastava-Owa fractional derivative operator [11], the Saitoh generalized linear operator, the Bernardi-Libera-Livingston operator and many others. One may

refer [11] for further details and references for these operators. Let  $H(n)$  denote the subclass of  $J(n)$  consisting of functions  $f$  of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0; n \in \mathbf{N}), \quad (1.10)$$

which are analytic and univalent in  $U$ .

**Definition 1.2.** Let  $K(A, B, \beta, n)$  consist of all analytic functions  $h$  in  $U$  for which

$$h(0) = 2$$

and

$$h(z) \prec \frac{1 + [B + (A - B)(1 - \beta)]z}{1 + Bz},$$

where  $-1 \leq B < A \leq 1, 0 < A \leq 1, 0 \leq \beta < 1$ .

**Definition 1.3.** For  $A, B$  fixed,  $-1 \leq B < A \leq 1, 0 \leq \beta < 1, a_j \in \mathbf{C}$  ( $j = 1, 2, \dots, q$ ),  $b_j \in \mathbf{C} \setminus \{0, -1, -2, \dots\}$  ( $j = 1, 2, \dots, s$ ),  $q \leq s + 1, q, s \in \mathbf{N}, z \in U$ , let  $H_n(A, B, \beta, q, s)$  denote the class of functions  $f \in H(n)$  of the form (1.10) for which

$$1 + \frac{z(H_s^q[a_1]f)''}{(H_s^q[a_1]f)'} \in K(A, B, \beta, n)$$

and

$$1 + \frac{z(H_s^q[a_1]f)''}{(H_s^q[a_1]f)'} \prec \frac{1 + [B + (A - B)(1 - \beta)]z}{1 + Bz}, \quad z \in U \quad (1.11)$$

where  $\prec$  denotes subordination.

From the definition, it follows that  $f \in H_n(A, B, \beta, q, s)$  if and only if there exists a function  $w(z)$  analytic in  $U$  and satisfies  $w(0) = 0$  and  $|w(z)| < 1$  for  $z \in U$ , such that

$$1 + \frac{z(H_s^q[a_1]f)''}{(H_s^q[a_1]f)'} = \frac{1 + [B + (A - B)(1 - \beta)]w(z)}{1 + Bw(z)}, \quad z \in U. \quad (1.12)$$

This condition (1.12) is equivalent to

$$\left| \frac{\frac{z(H_s^q[a_1]f)''}{(H_s^q[a_1]f)'}}{B + (A - B)(1 - \beta) - B \frac{z(H_s^q[a_1]f)''}{(H_s^q[a_1]f)'}} \right| < 1, \quad z \in U. \quad (1.13)$$

## 2. COEFFICIENT INEQUALITY

The following theorem gives a necessary and sufficient condition for function to be in the class  $H_n(A, B, \beta, q, s)$ .

**Theorem 2.1.** *Let the function  $f$  be defined by (1.10). Then the function  $f \in H_n(A, B, \beta, q, s)$  if and only if*

$$\sum_{k=n+1}^{\infty} Q(A, B, \beta) a_k \leq 1, \quad (2.1)$$

where

$$Q(A, B, \beta) = \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} \quad (2.2)$$

for

$$h(k) = \frac{(a_1)_{k-1} \cdots (a_q)_{k-1}}{(b_1)_{k-1} \cdots (b_s)_{k-1} (k-1)!} \quad (-1 \leq B < A \leq 1, 0 \leq \beta < 1, \\ a_j \in \mathbf{C} \quad (j = 1, 2, \dots, q), b_j \in \mathbf{C} \setminus \{0, -1, -2, \dots\} \quad (j = 1, 2, \dots, s)).$$

The result is sharp with the extremal function  $f$  given by

$$f(z) = z - \frac{1}{Q(A, B, \beta)} z^{n+1}, \quad (n \in \mathbf{N}). \quad (2.3)$$

*Proof.* Assume that the hypothesis (2.1) and  $|z| = 1$ , we note the following by the subordination property :

$$\begin{aligned} & |z(H_s^q[a_1]f)''| - |(B + (A - B)(1 - \beta))(H_s^q[a_1]f)' - Bz(H_s^q[a_1]f)''| = \left| - \sum_{k=n+1}^{\infty} k(k-1)h(k)a_k z^{k-1} \right| \\ & - \left| (B + (A - B)(1 - \beta)) \left( 1 - \sum_{k=n+1}^{\infty} kh(k)a_k z^{k-1} \right) + B \sum_{k=n+1}^{\infty} k(k-1)h(k)a_k z^{k-1} \right| \\ & \leq \sum_{k=n+1}^{\infty} k(k-1)h(k)a_k - (B + (A - B)(1 - \beta)) \\ & + \sum_{k=n+1}^{\infty} k(B + (A - B)(1 - \beta) - B(k-1))h(k)a_k \\ & = \sum_{k=n+1}^{\infty} k(B + (A - B)(1 - \beta) + (1 - B)(k-1)) \\ & h(k)a_k - (B + (A - B)(1 - \beta)) \leq 0. \end{aligned}$$

By hypothesis. Thus by maximum modulus Theorem  $f \in H_n(A, B, \beta, q, s)$ .

Conversely, suppose that  $f \in H_n(A, B, \beta, q, s)$ . Then by recalling the condition (1.13), we have

$$\begin{aligned} & \left| \frac{\frac{z(H_s^q[a_1]f)''}{(H_s^q[a_1]f)'} }{B + (A - B)(1 - \beta) - B \frac{z(H_s^q[a_1]f)''}{(H_s^q[a_1]f)'}} \right| = \left| \frac{z(H_s^q[a_1]f)''}{(B + (A - B)(1 - \beta))(H_s^q[a_1]f)' - Bz(H_s^q[a_1]f)''} \right| \\ & = \left| \frac{- \sum_{k=n+1}^{\infty} k(k-1)h(k)a_k z^{k-1}}{(B + (A - B)(1 - \beta)) \left( 1 - \sum_{k=n+1}^{\infty} kh(k)a_k z^{k-1} \right) + B \sum_{k=n+1}^{\infty} k(k-1)h(k)a_k z^{k-1}} \right| < 1. \end{aligned}$$

Since  $|Re(z)| \leq |z|$  for all  $z$ , we have

$$Re \left\{ \frac{\sum_{k=n+1}^{\infty} k(k-1)h(k)a_k z^{k-1}}{(B + (A-B)(1-\beta)) \left(1 - \sum_{k=n+1}^{\infty} kh(k)a_k z^{k-1}\right) + B \sum_{k=n+1}^{\infty} k(k-1)h(k)a_k z^{k-1}} \right\} < 1. \quad (2.4)$$

Now choosing values of  $z$  on the real axis and allowing  $z \rightarrow 1$  from the left through real values, the inequality (2.4) immediately yields the desired condition in (2.1). Finally, it is observed that the result is sharp and the extremal function is given by (2.3).  $\square$

Theorem 2.1 immediately yields the following result.

**Corollary 2.2.** *Let the function  $f$  of the form (1.10) is in the class  $H(n)$  and belongs to the class  $H_n(A, B, \beta, q, s)$ . Then*

$$a_k \leq \frac{B + (A-B)(1-\beta)}{k(B + (A-B)(1-\beta) + (1-B)(k-1)h(k))}, \quad (k \geq n+1, n \in \mathbf{N}), \quad (2.5)$$

where the equality holds true for the function (2.3).

*Proof.* The result (2.5) follows from the fact that the series in (2.1) converges.  $\square$

**Theorem 2.3.** *For  $n = 1$ , let  $f \in H_1(A, B, \beta, q, s)$  and  $g$  be an arbitrary element of  $H(1)$  such that  $g \prec f$ , defined in Definition 1.1, and if*

$$g_k = \frac{1}{k!} \left[ \frac{d^k(f(w(z)))}{dz^k} \right]_{z=0} \quad (2.6)$$

also if

$$\frac{\sum_{k=2}^{\infty} k(B + (A-B)(1-\beta) + (1-B)(k-1)h(k))|g_k|}{|g_1|} \leq B + (A-B)(1-\beta). \quad (2.7)$$

Then  $g \in H_1(A, B, \beta, q, s)$ .

*Proof.* Since  $g \prec f$  by definition of subordination there is analytic function  $w(z)$  such that  $|w(z)| \leq |z|$  and  $g(z) = f(w(z))$ . But  $g$  is the composition of two analytic functions in the unit disk, therefore we can expand this function in terms of Taylor series at origin as below

$$g(z) = \sum_{k=0}^{\infty} g_k z^k,$$

where  $g_k$  is defined in (2.6). Hence

$$g_0 = \frac{f(w(0))}{0!} = 0, \quad g_1 = \frac{w'(0)f'(0)}{1!} = w'(0).$$

Therefore, we can write

$$g(z) = g_1 z - \sum_{k=2}^{\infty} g_k z^k$$

and

$$(H_s^q[a_1]g)(z) = g_1 z - \sum_{k=2}^{\infty} h(k)g_k z^k,$$

we must prove  $g \in H_1(A, B, \beta, q, s)$ , in other words, we show that

$$\left| \frac{z((H_s^q[a_1]g)(z))''}{(B + (A - B)(1 - \beta))((H_s^q[a_1]g)(z))' - Bz((H_s^q[a_1]g)(z))''} \right| < 1,$$

or

$$\left| \frac{-\sum_{k=2}^{\infty} k(k-1)h(k)g_k z^{k-1}}{(B + (A - B)(1 - \beta))\left(g_1 - \sum_{k=2}^{\infty} kh(k)g_k z^{k-1}\right) + B\sum_{k=2}^{\infty} k(k-1)h(k)g_k z^{k-1}} \right| < 1.$$

Since  $|Re(z)| \leq |z|$  for all  $z$ , we have

$$Re \left\{ \frac{\sum_{k=2}^{\infty} k(k-1)h(k)g_k z^{k-1}}{(B + (A - B)(1 - \beta))\left(g_1 - \sum_{k=2}^{\infty} kh(k)g_k z^{k-1}\right) + B\sum_{k=2}^{\infty} k(k-1)h(k)g_k z^{k-1}} \right\} < 1. \quad (2.8)$$

We can choose value of  $z$  on the real axis so that  $((H_s^q[a_1]g)(z))'$  is real. Let  $z \rightarrow 1^-$  through real values, so we can write (2.8) as

$$\sum_{k=2}^{\infty} k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)g_k \leq g_1(B + (A - B)(1 - \beta)).$$

The proof is complete.  $\square$

### 3. SOME PROPERTIES OF A SUBCLASS $H_{1,c_m}(A, B, \beta, q, s)$

We introduce the class  $H_{1,c_m}(A, B, \beta, q, s)$  the subclass of  $H_1(A, B, \beta, q, s)$ , where

$$H_1(A, B, \beta, q, s) = \left\{ f \in H(1) : \left| \frac{\frac{z((H_s^q[a_1]f)(z))''}{((H_s^q[a_1]f)(z))'}}{B + (A - B)(1 - \beta) - B\frac{z((H_s^q[a_1]f)(z))''}{((H_s^q[a_1]f)(z))'}} \right| < 1 \right\}$$

consisting of functions with negative and fixed finitely many coefficients of the form:

$$f(z) = z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - \sum_{k=m+1}^{\infty} a_k z^k, \quad (3.1)$$

where  $m = 2, 3, \dots$ ,  $a_k \geq 0$  for  $k = m + 1, m + 2, \dots$ ,  $0 \leq c_i \leq 1$  for  $i = 2, 3, \dots, m$  and

$$0 \leq \sum_{i=2}^m c_i \leq 1.$$

The different cases were studied earlier by many authors e.g. [9], [10] and [13].

We need the following lemma which has been proved in general case in Theorem 2.1.

**Lemma 3.1.** *Let*

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in H(1).$$

Then  $f \in H_1(A, B, \beta, q, s)$  if and only if

$$\sum_{k=2}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)a_k}{B + (A - B)(1 - \beta)} \leq 1.$$

The following theorem gives a necessary and sufficient condition for a function to be in the class  $H_{1,c_m}(A, B, \beta, q, s)$ .

**Theorem 3.2.** *Let  $f$  be defined by (3.1). Then  $f \in H_{1,c_m}(A, B, \beta, q, s)$  if and only if*

$$\sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} a_k \leq 1 - \sum_{i=2}^m c_i. \quad (3.2)$$

*Proof.* By letting

$$a_i = \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)}.$$

Since  $H_{1,c_m}(A, B, \beta, q, s) \subset H_1(A, B, \beta, q, s)$  so  $f \in H_{1,c_m}(A, B, \beta, q, s)$  if and only if

$$\begin{aligned} & \sum_{i=2}^m \frac{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)}{(B + (A - B)(1 - \beta))} a_i \\ & + \sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} a_k \leq 1 \end{aligned}$$

or

$$\sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} a_k \leq 1 - \sum_{i=2}^m c_i$$

and this complete the proof.  $\square$

**Corollary 3.3.** *Let  $f$  defined by (3.1) be in the class  $H_{1,c_m}(A, B, \beta, q, s)$ . Then for  $k \geq m + 1$ , we have*

$$a_k \leq \frac{(B + (A - B)(1 - \beta)) \left(1 - \sum_{i=2}^m c_i\right)}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}.$$

This result is sharp due to the function  $f$  defined by

$$\begin{aligned} f(z) &= z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i \\ &\quad - \frac{(B + (A - B)(1 - \beta)) \left(1 - \sum_{i=2}^m c_i\right)}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)} z^k. \end{aligned}$$

**Theorem 3.4.** *Let*

$$f_i(z) = z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - \sum_{k=m+1}^{\infty} a_{k,j} z^k, \quad (3.3)$$

for  $j = 1, 2, \dots, \ell$  be in the class  $H_{1,c_m}(A, B, \beta, q, s)$ . Then the function

$$F(z) = \sum_{j=1}^{\ell} \eta_j f_j(z)$$

is also in the class  $H_{1,c_m}(A, B, \beta, q, s)$ , where

$$\sum_{j=1}^{\ell} \eta_j = 1, \quad 0 \leq c_i \leq 1, \quad 0 \leq \sum_{i=2}^m c_i \leq 1.$$

*Proof.* By Theorem 3.2 for every  $j = 1, 2, \dots, \ell$ , we have

$$\sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} a_{k,j} \leq 1 - \sum_{i=2}^m c_i.$$

But

$$F(z) = \sum_{j=1}^{\ell} \eta_j f_j(z)$$

$$= z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - \sum_{k=m+1}^{\infty} \left( \sum_{j=1}^{\ell} \eta_j a_{k,j} \right) z^k.$$

So

$$\begin{aligned} & \sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} \left( \sum_{j=1}^{\ell} \eta_j a_{k,j} \right) \\ &= \sum_{j=1}^{\ell} \sum_{k=m+1}^{\infty} \left( \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} a_{k,j} \right) \eta_j \\ &\leq \sum_{j=1}^{\ell} \left( 1 - \sum_{i=2}^m c_i \right) \eta_j = 1 - \sum_{i=2}^m c_i \end{aligned}$$

and the proof is complete.  $\square$

**Remark 3.1.** Let  $f_1, f_2$  be in the class  $H_{1,c_m}(A, B, \beta, q, s)$ . Then the function  $H(z) = \frac{1}{2}[f_1(z) + f_2(z)]$  is also in the class  $H_{1,c_m}(A, B, \beta, q, s)$ .

**Remark 3.2.** The class  $H_{1,c_m}(A, B, \beta, q, s)$  is a convex set.

In the next theorem, we will prove the arithmetic mean property.



**Theorem 3.5.** Let  $f_i, (j = 1, 2, \dots, \ell)$  define by (3.3) be in the class  $H_{1,c_m}(A, B, \beta, q, s)$ . Then the function

$$Q(z) = z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - \sum_{k=m+1}^{\infty} b_k z^k, \quad (b_k \geq 0)$$

is also in the class  $H_{1,c_m}(A, B, \beta, q, s)$ , where

$$b_k = \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j}.$$

*Proof.* We have

$$\begin{aligned} & \sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} b_k \\ &= \sum_{k=m+1}^{\ell} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} \left( \frac{1}{\ell} \sum_{j=1}^{\ell} \eta_j a_{k,j} \right) \\ &= \frac{1}{\ell} \sum_{j=1}^{\ell} \left( \sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} a_{k,j} \right) \\ & \quad \text{(by Theorem 3.2)} \\ &\leq \frac{1}{\ell} \sum_{j=1}^{\ell} \left( 1 - \sum_{i=2}^m c_i \right) = 1 - \sum_{i=2}^m c_i, \end{aligned}$$

and the proof is complete.  $\square$

**Definition 3.1.** Let  $f$  and  $g$  belong to  $H(n)$ . Then the weighted mean  $h_j(z)$  of  $f$  and  $g$  is given by  $h_j(z) = \frac{1}{2}[(1 - j)f(z) + (1 + j)g(z)]$ , where  $-1 \leq j \leq 1$ .

**Theorem 3.6.** Let  $f$  and  $g$  be in the class  $H_{1,c_m}(A, B, \beta, q, s)$ . Then the weighted mean of  $f$  and  $g$  is also in the class  $H_{1,c_m}(A, B, \beta, q, s)$ .

*Proof.* By using Definition 3.1, we obtain

$$\begin{aligned} & h_j(z) = \\ & \frac{1}{2} \left[ (1 - j) \left( z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - \sum_{k=m+1}^{\infty} a_k z^k \right) \right. \\ & \left. + (1 + j) \left( z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - \sum_{k=m+1}^{\infty} b_k z^k \right) \right] \end{aligned}$$

$$\begin{aligned}
&= z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i \\
&\quad - \sum_{k=m+1}^{\infty} \frac{1}{2} [(1 - j)a_k + (1 + j)b_k] z^k.
\end{aligned}$$

Since  $f$  and  $g$  are in the class  $H_{1,c_m}(A, B, \beta, q, s)$  using Theorem 3.4, we have

$$\begin{aligned}
&\sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} \frac{1}{2} [(1 - j)a_k + (1 + j)b_k] \\
&= \frac{1}{2} \sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} (1 - j)a_k \\
&\quad + \frac{1}{2} \sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{B + (A - B)(1 - \beta)} (1 + j)b_k \\
&\leq \frac{1}{2}(1 - j) \left( 1 - \sum_{j=2}^m c_j \right) + \frac{1}{2}(1 + j) \left( 1 - \sum_{i=2}^m c_i \right) = 1 - \sum_{i=2}^m c_i,
\end{aligned}$$

and again by Theorem 3.4,  $h_j(z) \in H_{1,c_m}(A, B, \beta, q, s)$ .  $\square$

Now, we obtain the extreme points of the class  $H_{1,c_m}(A, B, \beta, q, s)$  but we need the following theorem to prove

**Theorem 3.7.** *Let*

$$f_m(z) = z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i \quad (3.4)$$

and for  $k \geq m + 1$

$$\begin{aligned}
f_k(z) &= z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i \\
&\quad - \frac{(B + (A - B)(1 - \beta)) \left( 1 - \sum_{i=2}^m c_i \right)}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)} z^k.
\end{aligned} \quad (3.5)$$

Then the function  $Y$  is in the class  $H_{1,c_m}(A, B, \beta, q, s)$  if and only if it can be expressed in the form

$$Y(z) = \sum_{k=m}^{\infty} \sigma_k f_k(z),$$

where  $\sigma_k \geq 0$  ( $k \geq m$ ) and

$$\sum_{k=m}^{\infty} \sigma_k = 1.$$

*Proof.* Let

$$Y(z) = \sum_{k=m}^{\infty} \sigma_k f_k(z).$$

Then

$$\begin{aligned} Y(z) &= \sigma_m f_m(z) + \sum_{k=m+1}^{\infty} \sigma_k f_k(z) \\ &= \sigma_m z - \sigma_m \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i + \sum_{k=m+1}^{\infty} \sigma_k z \\ &\quad - \sum_{k=m+1}^{\infty} \sigma_k \left( \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i \right) \\ &\quad - \sum_{k=m+1}^{\infty} \sigma_k \left( \frac{(B + (A - B)(1 - \beta)) \left( 1 - \sum_{i=2}^m c_i \right)}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)} z^k \right) \\ &= \left( \sigma_m + \sum_{k=m+1}^{\infty} \sigma_k \right) z - \left( \sigma_m + \sum_{k=m+1}^{\infty} \sigma_k \right) \\ &\quad \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i \\ &\quad - \sum_{k=m+1}^{\infty} \frac{(B + (A - B)(1 - \beta)) \left( 1 - \sum_{i=2}^m c_i \right)}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)} \sigma_k z^k \\ &= z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i \\ &\quad - \sum_{k=m+1}^{\infty} \frac{(B + (A - B)(1 - \beta)) \left( 1 - \sum_{i=2}^m c_i \right) \sigma_k}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)} z^k. \end{aligned}$$

Finally, we have

$$\begin{aligned} &\sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1)) \left( 1 - \sum_{i=2}^m c_i \right) (B + (A - B)(1 - \beta)) \sigma_k h(k)}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)(B + (A - B)(1 - \beta))} \\ &= (1 - \sum_{i=2}^m c_i) \sum_{k=m+1}^{\infty} \sigma_k = (1 - \sum_{i=2}^m c_i) (1 - \sigma_m) \leq 1 - \sum_{i=2}^m c_i. \end{aligned}$$

Thus  $Y \in H_{1, c_m}(A, B, \beta, q, s)$ .

Conversely, assume  $Y \in H_{1,c_m}(A, B, \beta, q, s)$  so

$$Y(z) = z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - \sum_{k=m+1}^{\infty} a_k z^k.$$

By putting

$$\sigma_k = \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{(B + (A - B)(1 - \beta)) \left(1 - \sum_{i=2}^m c_i\right)} a_k, \quad (k \geq m + 1),$$

we have  $\sigma_k \geq 0$  and if we set

$$\sigma_m = 1 - \sum_{k=m+1}^{\infty} \sigma_k,$$

we get

$$\begin{aligned} Y(z) &= z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i \\ &\quad - \sum_{k=m+1}^{\infty} \frac{(B + (A - B)(1 - \beta)) \left(1 - \sum_{i=2}^m c_i\right) \sigma_k}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)} z^k \\ &= f_m(z) - \sum_{k=m+1}^{\infty} \left( z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - f_k(z) \right) \sigma_k \\ &= f_m(z) - \sum_{k=m+1}^{\infty} (f_m(z) - f_k(z)) \sigma_k \\ &= \left(1 - \sum_{k=m+1}^{\infty} \sigma_k\right) f_m(z) + \sum_{k=m+1}^{\infty} \sigma_k f_k(z) = \sum_{k=m}^{\infty} \sigma_k f_k(z). \end{aligned}$$

□

**Corollary 3.8.** *The extreme points of the class  $H_{1,c_m}(A, B, \beta, q, s)$  are the functions  $f_k$  ( $k \geq m$ ) defined by (3.4), (3.5).*

Now, we obtain the radii of starlikeness and convexity for the elements of the class  $H_{1,c_m}(A, B, \beta, q, s)$ .

**Theorem 3.9.** *Let the function  $f$  defined by (3.1) be in the class  $H_{1,c_m}(A, B, \beta, q, s)$ . Then  $f$  is starlike of order  $\eta$  ( $0 \leq \eta < 1$ ) in  $|z| < r$ , where  $r$  is the largest value such that*

$$\begin{aligned} &\sum_{i=2}^m \frac{c_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} r^{i-1} \\ &+ \frac{1 - \sum_{i=2}^m c_i}{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)} r^{k-1} \end{aligned}$$

$$\leq \frac{1}{(B + (A - B)(1 - \beta))}.$$

*Proof.* It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \eta. \quad (3.6)$$

Thus, we have

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &\leq \frac{\sum_{i=2}^m \frac{(i-1)(B+(A-B)(1-\beta))c_i}{i(B+(A-B)(1-\beta)+(1-B)(i-1))h(i)} |z|^{i-1} + \sum_{k=m+1}^{\infty} (k-1)a_k |z|^{k-1}}{1 - \sum_{i=2}^m \frac{(B+(A-B)(1-\beta))c_i}{i(B+(A-B)(1-\beta)+(1-B)(i-1))h(i)} |z|^{i-1} - \sum_{k=m+1}^{\infty} a_k |z|^{k-1}} \\ &\leq \frac{\sum_{i=2}^m \frac{(i-1)(B+(A-B)(1-\beta))c_i}{i(B+(A-B)(1-\beta)+(1-B)(i-1))h(i)} r^{i-1} + \sum_{k=m+1}^{\infty} (k-1) \frac{(B+(A-B)(1-\beta)) \left(1 - \sum_{i=2}^m c_i\right)}{k(B+(A-B)(1-\beta)+(1-B)(k-1))h(k)} r^{k-1}}{1 - \sum_{i=2}^m \frac{(B+(A-B)(1-\beta))c_i}{i(B+(A-B)(1-\beta)+(1-B)(i-1))h(i)} r^{i-1} - \sum_{k=m+1}^{\infty} \frac{(B+(A-B)(1-\beta)) \left(1 - \sum_{i=2}^m c_i\right)}{k(B+(A-B)(1-\beta)+(1-B)(k-1))h(k)} r^{k-1}} \end{aligned}$$

Therefore (3.6) holds true if the last term of above relationship is less than  $1 - \eta$  or equivalently

$$\begin{aligned} &\sum_{i=2}^m \frac{(i-\eta)(B+(A-B)(1-\beta))c_i}{i(B+(A-B)(1-\beta)+(1-B)(i-1))h(i)} r^{i-1} \\ &+ \frac{(k-\eta)(B+(A-B)(1-\beta)) \left(1 - \sum_{i=2}^m c_i\right)}{k(B+(A-B)(1-\beta)+(1-B)(k-1))h(k)} r^{k-1} \leq 1. \end{aligned}$$

Finally, we find

$$\begin{aligned} &\sum_{i=2}^m \frac{c_i}{i(B+(A-B)(1-\beta)+(1-B)(i-1))h(i)} r^{i-1} \\ &+ \frac{1 - \sum_{i=2}^m c_i}{k(B+(A-B)(1-\beta)+(1-B)(k-1))h(k)} r^{k-1} \leq \frac{1}{(B+(A-B)(1-\beta))} \end{aligned}$$

and this completes the proof.  $\square$

Making use of ( $f$  is convex function if and only if  $zf'$  is starlike function [2]), we obtain the following corollary :

**Corollary 3.10.** *Let  $f \in H_{1,c_m}(A, B, \beta, q, s)$ . Then  $f$  is convex of order  $\eta$  ( $0 \leq \eta < 1$ ) in  $|z| < r$ , where  $r$  is the largest value for which*

$$\begin{aligned} &\sum_{i=2}^m \frac{ic_i}{i(B+(A-B)(1-\beta)+(1-B)(i-1))h(i)} r^{i-1} \\ &+ \frac{k \left(1 - \sum_{i=2}^m c_i\right)}{k(B+(A-B)(1-\beta)+(1-B)(k-1))h(k)} r^{k-1} \leq \frac{1}{(B+(A-B)(1-\beta))} \end{aligned}$$

**Theorem 3.11.** Let  $f \in H_{1,c_m}(A, B, \beta, q, s)$  and

$$d_i = \frac{(B + (A - B)(1 - \beta))c_i^2}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)}, \quad (2 \leq i \leq m).$$

Then the function

$$h(z) = z - \sum_{i=2}^m \frac{(B + (A - B)(1 - \beta))d_i}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} z^i - \sum_{k=m+1}^{\infty} a_k z^k$$

is also in the class  $H_{1,c_m}(A, B, \beta, q, s)$ .

*Proof.* It can be verified that  $i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i) > 1, i = 2, 3, \dots, m$ . Therefore

$$0 \leq d_i = \frac{(B + (A - B)(1 - \beta))c_i^2}{i(B + (A - B)(1 - \beta) + (1 - B)(i - 1))h(i)} < c_i \leq 1.$$

So

$$0 \leq \sum_{i=2}^m d_i < \sum_{i=2}^m c_i \leq 1.$$

Thus

$$\begin{aligned} & \sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{(B + (A - B)(1 - \beta)) \left(1 - \sum_{i=2}^m d_i\right)} a_k \\ & \leq \sum_{k=m+1}^{\infty} \frac{k(B + (A - B)(1 - \beta) + (1 - B)(k - 1))h(k)}{(B + (A - B)(1 - \beta)) \left(1 - \sum_{i=2}^m c_i\right)} a_k \leq 1 \end{aligned}$$

and this completes the proof.  $\square$

#### REFERENCES

- [1] B. C. Carlson and D. B. Shaffer, *Starlike and pre- starlike hypergeometric functions*, SIAM J. Math. Anal.,15 (1984),737-745.
- [2] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer-Verlag, New York Berlin, Heidelberg, Tokyo, (1983).
- [3] J. Dziok and R. K. Raina, *Families of analytic functions associated with the wright generalized hypergeometric function*, Demonstratio Mathematica, 37(3) (2004), 533-542.
- [4] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Applied Mathematics and computation, 103(1) (1993), 1-13.
- [5] Yu. E. Hohlov, *Operators and operations in the class of univalent functions*, Izv. Vyss. Uceb. Zaved. Math., 10(1978), 83-89.
- [6] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., 28 (1981), 157-171.
- [7] S. S. Miller and P. T. Mocanu, *Differential subordinations: Theory and Applications*, Series on Monographs and Text Books in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

- [8] G. Murugusundaramoorthy and N. Magesh, *An application of second order differential inequalities based on linear and integral operators*, International J. of Math. Sci. and Engg. Appls., 2(1) (2008), 105-114.
- [9] M. Nunokawa, *A sufficient condition for univalence and starlikeness*, proc. Japan Acad. Ser. A Math. Sci., 65 (1989), 163-164.
- [10] S. Owa, M. Nunokawa and H. M. Srivastava, *A certain class of multivalent functions*, Appl. Math. Lett., 10(2) (1997), 7-10.
- [11] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., 39(5)(1987), 1057-1077.
- [12] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., 49 (1975), 109-115.
- [13] S. Shams and S. R. Kulkarni, *A class of univalent functions with negative and fixed finitely many coefficients*, Acta Cienica Indica, XXXIXM(3) (2003), 587-594.

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