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SUPERORDINATION RESULTS FOR MULTIVALENT FUNCTIONS INVOLVING A MULTIPLIER TRANSFORMATION

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ABSTRACT. The purpose of this paper is to derive superordination results involving a multiplier transformation for a family of analytic multivalent functions in the open unit disk.

1. INTRODUCTION

Let T_p denote the class of functions of the form :

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ (p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which are analytic and p - valent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)))$, for all $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence

$$f(z) \prec g(z), \ (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For $0 \leq \eta < p$, we denote by $S_p^*(\eta), K_p(\eta)$ and C_p the subclasses of T_p consisting of all analytic functions which are respectively, p - valent starlike of order η, p-valent convex of order η and close-to-convex in U.

Let define the multiplier transformation : $I_{\lambda,p}^s : T_p \to T_p$ by

$$I_{\lambda,p}^s f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^s a_k z^k, \ (\lambda \geq 0, s \in \mathbb{R}).$$

This operator is closely related to the Salagean derivative operator [13]. The special case $I_{1,\lambda}^s$ was studied recently by Cho and Srivastava [5] and Cho and Kim [4], while $I_{1,1}^s$ was studied by Uralegaddi and Somanatha [15]. An investigation of the $I_{p;\lambda}^s$ operator was given by Aghalary et. al. [1]. We also mention the papers [2], [7], [8], [9], [11], [12] and [14], that are closely-related recent articles on the subject of the multiplier transformations investigated in our work.

Setting

$$f_{p;\lambda}^s(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^s a_k z^k, \ (\lambda \geq 0, s \in \mathbb{R}),$$

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we now introduce the operator $I^s_{p;\lambda,\mu} : T_p \to T_p$, defined by

$$I^s_{p;\lambda,\mu} f(z) = f^s_{p;\lambda,\mu} * f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\lambda}{k+\lambda}\right)^s \frac{(p+\mu)_{k-p}}{(1)_{k-p}} a_k z^k, \tag{1.2}$$

where $f^s_{p;\lambda,\mu} f(z)$ is given by

$$f^s_{p;\lambda,\mu} f(z) * f^s_{p;\lambda}(z) = \frac{z^p}{(1-z)^{\mu+p}},\ (\mu > -p) \text{ (see [6])}. \tag{1.3}$$

In view of (1.2) and (1.3), we may easily obtain the following relations:

$$z(I^{s+1}_{p;\lambda,\mu} f(z))' = (\lambda+p) I^s_{p;\lambda,\mu} f(z) - \lambda I^{s+1}_{p;\lambda,\mu} f(z) \tag{1.4}$$

and

$$z(I^{s+1}_{p;\lambda,\mu} f(z))' = (\mu+p) I^s_{p;\lambda,\mu+1} f(z) - \mu I^s_{p;\lambda,\mu} f(z) \tag{1.5}$$

2. Preliminaries

To prove our results we shall need the following lemmas:

Lemma 2.1. *(see [10]) : Let $q(z)$ be convex univalent function in the unit disk U and $\gamma \in \mathbb{C}$. Further assume that $Re(\gamma) > 0$. If $r(z) \in H[q(0), 1] \cap Q$ and $r(z) + \gamma z r'(z)$ is univalent in U, then $q(z) + \gamma z q'(z) \prec r(z) + \gamma z r'(z)$, then $q(z) \prec r(z)$ and $q(z)$ is the best subordinant.*

Lemma 2.2. *(see [3]) : Let $q(z)$ be convex univalent function in the unit disk U, and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

: *(i) $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$.*

: *(ii) $zq'(z)\phi(q(z))$ is starlike univalent in $z \in U$.*

If $r(z) \in H[q(0), 1] \cap Q$, with $r(U) \subseteq D$ and if $\theta(r(z)) + zr'(z)\phi(r(z))$ is univalent in U and $\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(r(z)) + zr'(z)\phi(r(z))$, then $q(z) \prec r(z)$ and $q(z)$ is the best subordinant.

3. Main Results

Theorem 3.1. *Let q be convex univalent in U with $q(0) = 1$ and $\frac{1}{p^2} Re\beta > 0$. Let $f \in T_p$ satisfies $z^p I^{s+1}_{p;\lambda,\mu} f(z) \in H[q(0,1), 1] \cap Q$ and*

$$\left(1 + \frac{\beta}{p}\left(\frac{p-\lambda}{p}\right)\right) z^p \left(I^{s+1}_{p;\lambda,\mu} f(z)\right) + \left(\frac{\lambda\beta}{p}\left(\frac{p-1}{p}\right)\right) z^p \left(I^s_{p;\lambda,\mu} f(z)\right)$$

is univalent in U. If

$$q(z) + \frac{\beta}{p^2} z q'(z) \prec \left(1 + \frac{\beta}{p}\left(\frac{p-\lambda}{p}\right)\right) z^p \left(I^{s+1}_{p;\lambda,\mu} f(z)\right) + \left(\frac{\lambda\beta}{p}\left(\frac{p-1}{p}\right)\right) z^p \left(I^s_{p;\lambda,\mu} f(z)\right). \tag{3.1}$$

Then

$$q(z) \prec z^p (I^{s+1}_{p;\lambda,\mu} f(z)) \tag{3.2}$$

and $q(z)$ is the best subordinant of (3.1).

Proof. Let

$$r(z) = z^p (I^{s+1}_{p;\lambda,\mu} f(z)). \tag{3.3}$$

Differentiating (3.3) with respect to z, we get

$$\frac{zf'(z)}{r(z)} = \frac{(\lambda+p)(I^s_{p;\lambda,\mu} f(z)) - (\lambda+p)(I^{s+1}_{p;\lambda,\mu} f(z))}{(I^{s+1}_{p;\lambda,\mu} f(z))}. \tag{3.4}$$

From (3.4) and using the identity (1.4), a simple computation show that

$$\left(1+\frac{\beta}{p}\left(\frac{p-\lambda}{p}\right)\right)z^p(I^{s+1}_{p;\lambda,\mu}f(z))+\left(\frac{\lambda\beta}{p}\left(\frac{p-1}{p}\right)\right)z^p(I^{s}_{p;\lambda,\mu}f(z))=r(z)+\frac{\beta}{p^2}zr'(z)$$

and now, by using Lemma (2.1), we get desired result. □

Putting $q(z)=\left(\frac{1+z}{1-z}\right)^{\delta}, (0<\delta\leq 1)$ in the Theorem (3.1), we obtain the following Corollary :

Corollary 3.2. *Let* $0<\delta\leq 1$ *and* $\frac{1}{p^2}Re\beta>0$. *It* $z^pI^{s+1}_{p;\lambda,\mu}f(z)\in H[q(0),1]\cap Q$ *and*

$$\left(1+\frac{\beta}{p}\left(\frac{p-\lambda}{p}\right)\right)z^p(I^{s+1}_{p;\lambda,\mu}f(z))+\left(\frac{\lambda\beta}{p}\left(\frac{p-1}{p}\right)\right)z^p(I^{s}_{p;\lambda,\mu}f(z))$$

be univalent in U. *If*

$$\left(1+\frac{2\beta\delta z}{p^2(1-z^2)}\right)\left(\frac{1+z}{1-z}\right)^{\delta}$$
$$\prec\left(1+\frac{\beta}{p}\left(\frac{p-\lambda}{p}\right)\right)z^p\left(I^{s+1}_{p;\lambda,\mu}f(z)\right)+\left(\frac{\lambda\beta}{p}\left(\frac{p-1}{p}\right)\right)z^p\left(I^{s}_{p;\lambda,\mu}f(z)\right).$$

Then $\left(\frac{1+z}{1-z}\right)^{\delta}\prec z^p\left(I^{s+1}_{p;\lambda\mu}f(z)\right)$ *and* $q(z)$ *is the best subordinant.*

Theorem 3.3. *Let* q *be convex univalent in* U *with* $q(0)=1$ *and* $\frac{1}{1+p}Re\left(\frac{\alpha\lambda}{\delta}\right)>0$. *Let* $f\in T_p$ *satisfies :*

$$\frac{I^{s}_{p;\lambda,\mu}f(z)}{z^p}\in H[q(0,1),1]\cap Q$$

and

$$\frac{\alpha\lambda}{\delta(1+p)}\left[\frac{(\mu+p)(I^{s}_{p;\lambda,\mu+1}f(z))-(\mu+p)(I^{s}_{p;\lambda,\mu}f(z))}{z^p}\right]$$

is univalent in U. *If*

$$q(z)+\frac{\alpha\lambda}{\delta(1+p)}zq'(z)\prec\frac{\alpha\lambda}{\delta(1+p)}\left[\frac{(\mu+p)(I^{s}_{p;\lambda,\mu+1}f(z))-(\mu+p)(I^{s}_{p;\lambda,\mu}f(z))}{z^p}\right], \quad (3.5)$$

then

$$q(z)\prec\frac{I^{s}_{p;\lambda,\mu}f(z)}{z^p} \quad (3.6)$$

and $q(z)$ *is the best subordinant of (3.5).*

Proof. Let

$$f(z)=\frac{I^{s}_{p;\lambda,\mu}f(z)}{z^p}. \quad (3.7)$$

Differentiating (3.7) with respect to z, we get

$$\frac{zr'(z)}{r(z)}=\frac{(\mu+p)(I^{s}_{p;\lambda,\mu+1}f(z))}{(I^{s}_{p;\lambda,\mu}f(z))}-(\mu+p). \quad (3.8)$$

From (3.8) and using the identity (1.5), a simple computation shows that

$$\frac{\alpha\lambda}{\delta(1+p)}\left[\frac{(\mu+p)(I^{s}_{p;\lambda,\mu+1}f(z))-(\mu+p)(I^{s}_{p;\lambda,\mu}f(z))}{z^p}\right]=r(z)+\frac{\alpha\lambda}{\delta(1+p)}zr'(z)$$

and now, by using Lemma (2.1), we get desired result. □

Putting $q(z)=\frac{1+Az}{1+Bz}$ in the Theorem (3.2) , we have the following Corollary :

Corollary 3.4. *Let $A, B \in \mathbb{C}, A \neq B, |B| < 1$. If $f \in T_p$:*

$$\frac{I^s_{p;\lambda,\mu} f(z)}{z^p} \in H[q(0), 1] \cap Q$$

and

$$\frac{\alpha\lambda}{\delta(1+p)} \left[\frac{(\mu+p)(I^s_{p;\lambda,\mu+1} f(z)) - (\mu+p)(I^s_{p;\lambda,\mu} f(z))}{z^p} \right]$$

is univalent in U. If

$$\frac{1+Az}{1+Bz} + \frac{\alpha\lambda}{\delta(1+p)} \frac{(A-B)z}{(1+Bz)^2} \prec \frac{\alpha\lambda}{\delta(1+p)} \left[\frac{(\mu+p)(I^s_{p;\lambda,\mu+1} f(z)) - (\mu+p)(I^s_{p;\lambda,\mu} f(z))}{z^p} \right].$$

Then $\left(\frac{1+Az}{1+Bz}\right) \prec \frac{I^s_{p;\lambda,\mu} f(z)}{z^p}$ and $q(z) = \frac{1+Az}{1+Bz}$ is the best subordinant.

Theorem 3.5. *Let $\alpha_i \in \mathbb{C}, (i = 1, 2, 3)$ and let q be convex univalent with $q(0) = 1$, and assume that*

$$Re\left\{ \frac{\alpha_2}{\alpha_3} q'(z) q(z) \right\} > 0. \tag{3.9}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $f \in T_p$ satisfies :

$$z^p I^{s+1}_{p;\lambda,\mu} f(z) \in H[q(0), 1] \cap Q$$

and

$$\alpha_1 + \alpha_2 z^p (I^{s+1}_{p;\lambda,\mu} f(z)) + \alpha_3 \left[\frac{(\lambda+p)(I^s_{p;\lambda,\mu} f(z))}{(I^{s+1}_{p;\lambda,\mu} f(z))} - (\lambda - p) \right]$$

is univalent in U. If

$$\alpha_1 + \alpha_2 q(z) + \alpha_3 \left[\frac{zq'(z)}{q(z)} \right] \prec \alpha_1 + \alpha_2 z^p (I^{s+1}_{p;\lambda,\mu} f(z)) + \alpha_3 \left[\frac{(\lambda+p)(I^s_{p;\lambda,\mu} f(z))}{(I^{s+1}_{p;\lambda,\mu} f(z))} - (\lambda - p) \right], \tag{3.10}$$

then

$$q(z) \prec z^p (I^{s+1}_{p;\lambda,\mu} f(z)) \tag{3.11}$$

and q is the best subordinant of (3.10).

Proof. Define the function $r(z)$ by

$$r(z) = z^p (I^s_{p;\lambda,\mu}(f(z)). \tag{3.12}$$

By setting $\theta(w) = \alpha_1 + \alpha_2 w$ and $\phi(w) = \frac{\alpha_3}{w}$, we see that $\theta(w)$ is analytic in $\mathbb{C}, \phi(w)$ is analytic in $\mathbb{C}/\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}/\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = z\frac{\alpha_3}{q(z)} q'(z).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{ \frac{z\theta'(q(z))}{\phi(q(z))} \right\} = Re\left\{ \frac{\alpha_2}{\alpha_3} q(z) q'(z) \right\} > 0.$$

By a straight forward computation, we obtain

$$\alpha_1 + \alpha_2 z^p (I^{s+1}_{p;\lambda,\mu} f(z)) + \alpha_3 \left[\frac{(\lambda+p)(I^s_{p;\lambda,\mu} f(z))}{(I^{s+1}_{p;\lambda,\mu} f(z))} - (\lambda - p) \right] \prec \alpha_1 + \alpha_2 r(z) + \alpha_3 \left[\frac{zr'(z)}{r(z)} \right]. \tag{3.13}$$

From (3.10) and (3.13), we have

$$\alpha_1+\alpha_2 q(z)+\alpha_3\left[\frac{zq'(z)}{q(z)}\right]\prec\alpha_1+\alpha_2 r(z)+\alpha_3\left[\frac{zr'(z)}{r(z)}\right]. \tag{3.14}$$

Therefore, by Lemma (2.2), we get $q(z)\prec r(Z)$. □

Putting $q(z)=\frac{1+Az}{1+Bz}$ in the Theorem (3.3), we obtain the following corollary.

Corollary 3.6. *Let* $\alpha_i\in\mathbb{C},(i=1,2,3)$ *and let be convex univalent with* $q(0)=1$ *and assume that :*

$$Re\left\{\frac{\alpha_2}{\alpha_3}\left(\frac{1+Az}{1+Bz}\right)\left(\frac{A-B}{(1+Bz)^2}\right)\right\}>0.$$

Suppose that $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U*. Let* $f\in T_p$ *satisfies :*

$$z^pI^{s+1}_{p;\lambda,\mu}f(z)\in H[q(0),1]\cap Q$$

and

$$\alpha_1+\alpha_2 z^p(I^{s+1}_{p;\lambda,\mu}f(z))+\alpha_3\left[\frac{(\lambda+p)(I^{s}_{p;\lambda,\mu}f(z))}{(I^{s+1}_{p;\lambda,\mu}f(z))}-(\lambda-p)\right]$$

is univalent in U*. If*

$$\alpha_1+\alpha_2\left(\frac{1+Az}{1+Bz}\right)+\alpha_3\left[\frac{z(A-B)}{(1+Az)/(1+Bz)}\right]$$
$$\prec\alpha_1+\alpha_2 z^p(I^{s+1}_{p;\lambda,\mu}f(z))+\alpha_3\left[\frac{(\lambda+p)(I^{s}_{p;\lambda,\mu}f(z))}{(I^{s+1}_{p;\lambda,\mu}f(z))}-(\lambda-p)\right].$$

Then $\left(\frac{(1+Az}{1+Bz}\right)\prec z^p(I^{s+1}_{p;\lambda,\mu}f(z))$ *and* $q(z)=\left(\frac{1+Az}{1+Bz}\right)$ *is the best subordinant.*

Theorem 3.7. *Let* $\sigma,\delta,\gamma\in\mathbb{C}$ *and* q *be convex univalent in* U *with* $q(0)=1$ *and assume that :*

$$Re\left\{\frac{\sigma q'(z)}{\gamma}\right\}>0. \tag{3.15}$$

Suppose that $zq'(z)$ *is starlike univalent in* U*. Let* $f\in T_p$ *satisfies :*

$$z^pI^{s+1}_{p;\lambda,\mu}f(z)\in H[q(0),1]\cap Q$$

and

$$(\sigma-\lambda\gamma+\gamma p)z^p(I^{s+1}_{p;\lambda,\mu}f(z))+\gamma(\lambda+p)(I^{s}_{p;\lambda,\mu}f(z))+\delta$$

is univalent in U*. If*

$$\sigma q(z)+\gamma zq'(z)+\delta\prec(\sigma-\lambda\gamma+\gamma p)z^p(I^{s+1}_{p;\lambda,\mu}f(z))+\gamma(\lambda+p)(I^{s}_{p;\lambda,\mu}f(z))+\delta, \tag{3.16}$$

then

$$q(z)\prec z^p(I^{s+1}_{p;\lambda,\mu}f(z)) \tag{3.17}$$

and $q(z)$ *is the best subordinant of (3.17).*

Proof. Define the function $r(z)$ by

$$r(z)=z^p(I^{s}_{p;\lambda,\mu}f(z)). \tag{3.18}$$

By setting $\theta(w)=\sigma w+\delta$ and $\phi(w)=\gamma$, we see that $\theta(w)$ is analytic in $\mathbb{C},\phi(w)$ is analytic in $\mathbb{C}/\{0\}$ and that $\phi(w)\neq 0,w\in\mathbb{C}/\{0\}$. Also, we get

$$Q(z)=zq'(z)\phi(q(z))=\gamma zq'(z).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = Re\left\{\frac{\sigma q'(z)}{\gamma}\right\} > 0.$$

By a straight forward computation, we obtain

$$(\sigma - \lambda\gamma + \gamma p)z^p(I^{s+1}_{p;\lambda,\mu}f(z)) + \gamma(\lambda + p)(I^s_{p;\lambda,\mu}f(z)) + \delta \prec \sigma r(z) + \gamma z r'(z) + \delta. \tag{3.19}$$

From (3.16) and (3.19) we have

$$\sigma q(z) + \gamma z q'(z) + \delta \prec \sigma r(z) + \gamma z r'(z) + \delta.$$

Therefore, by Lemma (2.2), we get $q(z) \prec r(z)$. □

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