Analele Universităţii Oradea
Fasc. Matematica, Tom XXI (2014), Issue No. 1, 145–152

SOME RESULTS OF SECOND ORDER DIFFERENTIAL SUBORDINATION FOR FRACTIONAL INTEGRAL OF DZIOK-SRIVASTAVA OPERATOR

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ABSTRACT. Using a fractional integral of Dziok-Srivastava operator, we introduce and study some differential subordination results for analytic functions in the open unit disk. These results are obtained by investigating classes of admissible functions. Some of the results in this paper would provide extensions of those given in earlier works.

1. INTRODUCTION

Let $H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, n]$ be the subclass of consisting of functions of the form $f(z) = a+a_n z^n+a_{n+1}z^{n+1}+\cdots$, with $H_0 = [0, 1]$ and $H_1 = H[1, 1]$. We denote by the $\mathcal{A}$ class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ (z \in U, \ n \in \mathbb{N}), \tag{1.1}$$

which are analytic in U. Let f and g members of $H(U)$. The function is said to be subordinate to g in U, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ is analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ in U, such that $f(z) = g(w(z))$ $(z \in U)$ (see [1] and [7]).
In particular, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.2}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply dominant if $p \prec q$ for all the function p satisfies (1.2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant of (1.2).

Definition 1.1. (see [4]) The fractional integral of order λ $(\lambda > 0)$ is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 30C45, 30C80, 33C10.
Key words and phrases. Analytic function, Differential subordination, Univalent function, Fractional integral, Dziok-Srivastava operator.

where f is analytic function in a simply-connected region of z-plane containing the origin, and the multiplicity of $(z-t)^{1-\lambda}$ is removed by requiring $\log(z-t)$ to be real, when $Re(z-t) > 0$.

Definition 1.2. (see [3]) For $f \in \mathcal{A}$, the Dziok-Srivastava operator is defined by

$$H_m^l(\alpha_1, \alpha_2, \cdots, \alpha_l; \beta_1, \beta_2, \cdots, \beta_m)f(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\cdots w(\beta_m)_{n-1}(n-1)!} a_n z^n, \tag{1.4}$$

$\alpha_i \in \mathbb{C}, i = 1, 2, \cdots, l; \beta_j \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}, j = 1, 2, \cdots, m$; where $(x)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)(x+2)\cdots(x+n-1) & \text{for } n \in N \text{ and } x \in \mathbb{C}, \\ 1 & \text{if } n = 0 \text{ and } x \in \mathbb{C} \setminus \{0\}. \end{cases}$$

For simplicity, we write $H_m^l[\alpha_1]f(z) = H_m^l(\alpha_1, \alpha_2, \cdots, \alpha_l; \beta_1, \beta_2, \cdots, \beta_m)f(z)$.

From Definition 1.1 and Definition 1.2, we get the fractional integral of Dziok-Srivastava operator:

$$D_z^{-\lambda} H_m^l[\alpha_1]f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)(\alpha_1)_{n-1}(\alpha_2)_{n-1}\cdots(\alpha_l)_{n-1}}{\Gamma(n+1+\lambda)(\beta_1)_{n-1}(\beta_2)_{n-1}\cdots(\beta_m)_{n-1}(n-1)!} a_n z^{n+\lambda}. \tag{1.5}$$

We note from (1.5) that, we have

$$z(D_z^{-\lambda} H_m^l[\alpha_1]f(z))' = \alpha_1 D_z^{-\lambda} H_m^l[\alpha_1+1]f(z) - [\alpha_1 - (1+\lambda)] D_z^{-\lambda} H_m^l[\alpha_1]f(z). \tag{1.6}$$

In order to prove the results, we shall need the following Definition and Theorem.

Definition 1.3. [7, Definition 2.2b p. 21] Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that for $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of Q for which $q(0) = a$ be denoted by $Q(a), Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Definition 1.4. [7, Definition 2.3a p.27] Let Ω be a set in $\mathbb{C}, q \in Q$ and let n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$, whenever $r = q(\zeta), s = k\zeta q'(\zeta)$ and $Re\left\{\frac{t}{s} + 1\right\} \geq k\ Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}$, $z \in U$, $\zeta \in \partial U \setminus E(q)$, $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\overline{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M = \{w : |w| < M\}, q(0) = a, E(q) = \phi$ and $q \in Q(a)$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$ and in the special case when $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, \alpha]$.

Theorem 1.1. [7, Theorem 2.3 b3, p. 28] *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots (z \in U)$ satisfies the following inclusion relationship $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$, then $p(z) \prec q(z)$ $(z \in U)$.*

Similar results have been obtained for other classes of functions in [2, 5, 6].

2. Subordination Results

Definition 2.1. Let Ω be a set in $\mathbb{C}, q(z) \in Q_1 \cap H[q(0), 1]$. The class of admissible functions $\Phi_n[\Omega, q]$ consists of those function $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; z) \notin \Omega$, whenever

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + [\alpha_1 - \lambda]q(\zeta)}{\alpha_1}$$

$$Re\left\{\frac{\alpha_1(\alpha_1+1)w - (\alpha_1-\lambda)[\alpha_1-\lambda+1]u}{\alpha_1 v - [\alpha_1-\lambda]u} - 2(\alpha_1 - \lambda + \frac{1}{2})\right\} \geq kRe\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$z \in U$, $\zeta \in \partial U \setminus E(q)$, $k \geq 1$.

Theorem 2.1. *Let $\phi \in \Phi_n[\Omega, q]$. If $f(z) \in \mathcal{A}$ satisfies*

$$\left\{\phi((D_z^{-\lambda}H_m^l[\alpha_1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'; z) : z \in U\right\} \subset \Omega \tag{2.1}$$

then $(D_z^{-\lambda}H_m^l[\alpha_1]f(z))' \prec q(z)$.

Proof. Define the analytic function $g(z)$ in U by

$$g(z) = (D_z^{-\lambda}H_m^l[\alpha_1]f(z))'. \tag{2.2}$$

In view of relation

$$z(D_z^{-\lambda}H_m^l[\alpha_1]f(z))' = \alpha_1 D_z^{-\lambda}H_m^l[\alpha_1+1]f(z) - [\alpha_1 - (1+\lambda)]D_z^{-\lambda}H_m^l[\alpha_1]f(z), \tag{2.3}$$

from (2.2) we have

$$(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))' = \frac{zg'(z) + [\alpha_1-\lambda]g(z)}{\alpha_1}. \tag{2.4}$$

Further, a simple computation shows that

$$(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))' = \frac{z^2g''(z) + 2[\alpha_1-\lambda+1]zg'(z) + (\alpha_1-\lambda)[\alpha_1-\lambda+1]g(z)}{\alpha_1(\alpha_1+1)} \tag{2.5}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r, s, t) = r,\ v(r, s, t) = \frac{s + [\alpha_1-\lambda]r}{\alpha_1},$$

$$w(r, s, t) = \frac{t + 2[\alpha_1-\lambda+1]s + (\alpha_1-\lambda)[\alpha_1-\lambda+1]r}{\alpha_1(\alpha_1+1)}. \tag{2.6}$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi\left(r, \frac{s+[\alpha_1-\lambda]r}{\alpha_1}, \frac{t + 2[\alpha_1-\lambda+1]s + (\alpha_1-\lambda)[\alpha_1-\lambda+1]r}{\alpha_1(\alpha_1+1)}; z\right). \tag{2.7}$$

The proof shall make use of Theorem 1.1. Using (2.2), (2.4) and (2.5), from (2.7), we obtain

$$\psi(g(z), zg'(z), z^2g''(z); z) = \phi((D_z^{-\lambda}H_m^l[\alpha_1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'; z). \tag{2.8}$$

Hence (2.1) becomes

$$\begin{aligned}&\phi((D_z^{-\lambda}H_m^l[\alpha_1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'; z)\\ &= \psi(g(z), zg'(z), z^2g''(z); z) \in \Omega.\end{aligned} \tag{2.9}$$

Note that

$$\frac{t}{s} + 1 = \frac{\alpha_1(\alpha_1+1)w + (\alpha_1-\lambda)[\alpha_1-\lambda+1]u}{\alpha_1 v - [\alpha_1-\lambda]u} - 2\left(\alpha_1 - \lambda + \frac{1}{2}\right),$$

and since the admissibility condition for $\phi \in \Phi_n[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.4, hence $\phi \in \Psi[\Omega, q]$, and by Theorem 1.1, then $g(z) \prec q(z)$ or $D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q(z)$. □

Theorem 2.2. *Let $\phi \in \Phi_n[h, q]$ with $q(0) = 1$. If $f(z) \in \mathcal{A}$ satisfies*

$$\phi((D_z^{-\lambda} H_m^l[\alpha_1] f(z))', (D_z^{-\lambda} H_m^l[\alpha_1 + 1] f(z))', (D_z^{-\lambda} H_m^l[\alpha_1 + 2] f(z))'; z) \prec h(z), \tag{2.10}$$

then $(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q(z)$ $(z \in U)$.

Our next result is an extension of Theorem 2.1 to the case the behavior of $q(z)$ on ∂U is not known.

Corollary 2.3. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in U with $q(0) = 1$. Let $\phi \in \Phi_n[\Omega, q_p]$ for some $p \in (0, 1)$ where $q_p(z) = q(pz)$. If $f(z) \in \mathcal{A}$ and $\phi((D_z^{-\lambda} H_m^l[\alpha_1] f(z))', (D_z^{-\lambda} H_m^l[\alpha_1 + 1] f(z))', D_z^{-\lambda} H_m^l[\alpha_1 + 2] f(z))'; z) \in \Omega$, then $(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q(z)$ $(z \in U)$.*

Proof. By Theorem 2.1, we get $(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q_p(z)$. □

The result is now deduced from the following subordination relationship $q_p(z) \prec q(z)$ $(z \in U)$.

Theorem 2.4. *Let $h(z)$ and $q(z)$ be univalent in U with $q(0) = 1$ and set $q_p(z) = q(pz)$ and $h_p(z) = h(pz)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$, satisfy one of the following conditions:*
(1) $\phi \in \Phi_n[h, q_p]$, for some $p \in (0, 1)$, or
(2) there exists $p_0 \in (0, 1)$ such that $\phi \in \Phi_n[h_p, q_p]$, for all $p \in (p_0, 1)$.
If $f(z) \in \mathcal{A}$ satisfy (2.10), then $(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q(z)$ $(z \in U)$.

Proof. (1) By applying Theorem 2.1, we obtain $(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q_p(z)$. Since $q_p(z) \prec q(z)$, we deduce $(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q(z)$.

(2) If we let $g_p(z) = (D_z^{-\lambda} H_m^l[\alpha_1] f_p(z))' = \left(D_z^{-\lambda} H_m^l[\alpha_1] f(pz)\right)' = g(pz)$, then

$$\begin{aligned} &\phi((D_z^{-\lambda} H_m^l[\alpha_1] f_p(z)))' = \left(D_z^{-\lambda} H_m^l[\alpha_1 + 1] f_p(z)\right)', \left(D_z^{-\lambda} H_m^l[\alpha_1 + 2] f_p(z)\right)'; pz) \\ = \; &\phi((D_z^{-\lambda} H_m^l[\alpha_1] f(pz))', \left(D_z^{-\lambda} H_m^l[\alpha_1 + 1] f(pz)\right)', \left(D_z^{-\lambda} H_m^l[\alpha_1 + 2] f(pz)\right)'; pz) \in h_p(U). \end{aligned}$$

By using Theorem 2.1 and the comment associated with (2.9) with $w(z) = pz$, we obtain $(D_z^{-\lambda} H_m^l[\alpha_1] f_p(z))' \prec q_p(z)$, for $p \in (p_0, 1)$. By letting $p \to 1^-$, we obtain $(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q(z)$. The next result give the best dominant of the differential subordination (2.10). □

Theorem 2.5. *Let $h(z)$ be univalent in U and $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose the differential equation*

$$\phi\left(q(z), \frac{zq'(z)|[\alpha_1 - \lambda]q(z)}{\alpha_1}, \frac{z^2 q''(z) + 2[\alpha_1 - \lambda + 1]zq'(z) + (\alpha_1 - \lambda)[\alpha_1 - \lambda + 1]q(z)}{\alpha_1(\alpha_1 + 1)}; z\right) = h(z) \tag{2.11}$$

has a solution $q(z)$ with $q(0) = 1$ and satisfy one of the following conditions:
(1) $q(z) \in Q$ and $\phi \in \Phi_n[h, q]$
(2) $q(z)$ is univalent in U and $\phi \in \Phi_n[h, q_p]$ for some $p \in (0, 1)$, or
(3) $q(z)$ is univalent in U and there exists $p_0 \in (0, 1)$ such that $\phi \in \Phi_n[h_p, q_p]$ for all $p \in (p_0, 1)$.
If $f(z) \in \mathcal{A}$ satisfy (2.10), then $(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' \prec q(z)$ $(z \in U)$ and $q(z)$ is the best dominant.

Proof. By applying Theorem 2.2 and 2.4, we deduce that $q(z)$ is a dominant of (2.10). Since $q(z)$ satisfies (2.11), it is also a solution of (2.10) and therefore $q(z)$ will be dominated by all dominants of (2.10). Hence $q(z)$ is the best dominant of (2.10). □

In the particular case $q(z) = 1 + Mz, M > 0$ and in view of Definition 2.1, the class of admissible function $\Phi_n[\Omega, q]$, denoted by $\Phi_n[\Omega, M]$, is described below.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_n[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that:

$$\phi\left(1 + Me^{i\theta}, 1 + \tfrac{[k+\alpha_1-\lambda]Me^{i\theta}-\lambda}{\alpha_1}, 1 + \tfrac{L+\lambda(\lambda-2\alpha_1-1)+[2[\alpha_1-\lambda+1]k+(\alpha_1-\lambda)[\alpha_1-\lambda+1]]Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right) \notin \Omega, \tag{2.12}$$

whenever $z \in U, k \geq nM, Re(Le^{-i\theta}) \geq (n-1)k$ and $\theta \in R$.

From above definition and Theorem 2.1, we have

Corollary 2.6. *Let $\phi \in \Phi_n[\Omega, M]$. If $f(z) \in \mathcal{A}$ satisfy the following inclusion relationship $\phi((D_z^{-\lambda}H_m^l[\alpha_1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'; z) \in \Omega$, then $(D_z^{-\lambda}H_m^l[\alpha_1]f(z))' \prec 1 + Mz$ $(z \in U)$.*

In the special case $\Omega = q(U) = \{w : |w-1| < M\}$, the class $\Phi_n[\Omega, M]$is simply denoted by $\Phi_n[M]$.

Corollary 2.7. *Let $\phi \in \Phi_n[M]$. If $f(z) \in \mathcal{A}$ satisfies the following inclusion relationship $|\phi((D_z^{-\lambda}H_m^l[\alpha_1]f(z))], (D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'; z) - 1| < M$, then $|(D_z^{-\lambda}H_m^l[\alpha_1]f(z))' - 1| < M$.*

Corollary 2.8. *If $M > 0$ and $f(z) \in \mathcal{A}$ satisfies the following inclusion relationship $|(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))', (D_z^{-\lambda}H_m^l[\alpha_1]f(z))'| < \frac{M}{\alpha_1}$, then $|(D_z^{-\lambda}H_m^l[\alpha_1]f(z))' - 1| < M$.*

Proof. This follows from Corollary 2.6 by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(z)$ where $h(z) = \frac{M}{\alpha_1}z$, $(M > 0)$. To use the Corollary 2.6 we need to show that $\phi \in \Phi_n[\Omega, M]$, that is the admissibility condition (2.12) is satisfied. This follows since

$$\begin{aligned} &\left|\phi\left(1 + Me^{i\theta}, 1 + \frac{[k+\alpha_1-\lambda]Me^{i\theta}-\lambda}{\alpha_1},\right.\right. \\ &\left.\left. 1 + \frac{L+\lambda(\lambda_2-2\alpha_1-1)+[2[\alpha_1-\lambda+1]k+(\alpha_1-\lambda)[\alpha_1-\lambda+1]]Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z\right)\right| \\ = &\left|\frac{\left((k-\lambda)\frac{\lambda}{Me^{i\theta}}\right)Me^{i\theta}}{\alpha_1}\right| = \frac{M}{\alpha_1}\left|k - \frac{\lambda(1+M)}{\alpha_1}\right| \geq \frac{M}{\alpha_1}, \end{aligned}$$

whenever $z \in U, k \geq nM$, $Re(Le^{-i\theta}) \geq (n-1)k$ and $\theta \in R$. □

The required result now follows from Corollary 2.6. Theorem 2.5 shows that the result is sharp. the differential equation $\frac{zq'(z)}{\alpha_1} = \frac{M}{\alpha_1}z$ $(\alpha_1 < M)$ has a univalent solution $q(z) = 1 + Mz$. It follows from Theorem 2.5 that $q(z) = 1 + Mz$ is the best dominant.

Definition 2.3. Let Ω be a set in $\mathbb{C}$, $q(z) \in Q \cap H[q(0), 1]$. The class of admissible functions $\Phi_{n,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, , v, w; z) \notin \Omega$, whenever

$$u = q(\zeta),\ v = \frac{q(\zeta)(\alpha_1 q(\zeta)+1) + k\zeta q'(\zeta)}{(\alpha_1+1)q(\zeta)},\ (q(\zeta) \neq 0),$$

$$\begin{aligned} Re&\left\{\frac{(\alpha_1+1)v[[\alpha_1+2-\lambda] + (\alpha_1+1)v - (\alpha_1+2)w]}{[\alpha_1 u + 1] - (\alpha_1+1)v} - [[2\alpha_1 u + 1] - (\alpha_1+1)v]\right\} \\ &\geq kRe\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}, \quad z \in U, \zeta \in \partial U \setminus E(q), k \geq 1. \end{aligned}$$

Theorem 2.9. *Let* $\phi \in \Phi_{n,1}[\Omega, q]$. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left\{\phi\left(\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+3]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}; z\right) : z \in U\right\} \subset \Omega, \tag{2.13}$$

then $\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'} \prec q(z)$.

Proof. Define the analytic function $g(z)$ in U by

$$g(z) = \frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'}. \tag{2.14}$$

Then, by using (2.3), we get

$$\frac{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'} = \frac{g(z)(\alpha_1 g(z)+1) + zg'(z)}{(\alpha_1+1)g(z)}. \tag{2.15}$$

Differentiating logarithmically (2.15) , and further computations show that

$$\frac{(D_z^{-\lambda}H_m^l[\alpha_1+3]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'} = \frac{1}{(\alpha_1+2)} \times \left\{\alpha_1(g(z)+1) + (3-\lambda) + \frac{zg'(z)}{g(z)} + \frac{\alpha_1 g'(z) + \frac{zg'(z)}{g(z)} + \frac{z^2g''(z)}{g(z)} - \left(\frac{zg'(z)}{g(z)}\right)^2}{(\alpha_1 g(z)+1) + \frac{zg'(z)}{g(z)}}\right\}. \tag{2.16}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{aligned} u(r,s,t) = {} & r, \;\; v(r,s,t) = \frac{r(\alpha_1 r+1)+s}{(\alpha_1+1)r}, \\ w(r,s,t) = {} & \frac{1}{(\alpha_1+2)}\left(\alpha_1(r+1) + (3-\lambda) + \frac{s}{r} + \frac{a_1 s + \frac{s}{r} + \frac{t}{r} - \left(\frac{s}{r}\right)^2}{(\alpha_1 r+1)+\frac{s}{r}}\right). \end{aligned} \tag{2.17}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{r(\alpha_1 r+1)+s}{(\alpha_1+1)r}, \frac{1}{(\alpha_1+2)}\left(\alpha_1(r+1) + (3-\lambda) + \frac{s}{r} + \frac{a_1 s + \frac{s}{r} + \frac{t}{r} - \left(\frac{s}{r}\right)^2}{(\alpha_1 r+1)+\frac{s}{r}}\right); z\right). \tag{2.18}$$

Using (2.14), (2.15) and (2.16), from (2.18), it follows that

$$\begin{aligned} & \psi(g(z), zg'(z), z^2g''(z); z) \\ = {} & \phi\left(\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+3]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}; z\right). \end{aligned}$$

Hence (2.13) $\psi(g(z), zg'(z), z^2g''(z); z) \in \Omega$. The proof is complete if it can be shown that the admissibility condition for $\phi \in \Phi_{n,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.4. For this purpose, note that

$$\begin{aligned} \frac{s}{r} &= (\alpha_1+1)v - [\alpha_1 u + 1], \\ \frac{t}{r} &= [(\alpha_1+2)w - [\alpha_1+2-\lambda]](\alpha_1+1)v - (\alpha_1+1)^2v^2 - \left[(\alpha_1+1) - 2\frac{s}{r}\right]\frac{s}{r'}, \end{aligned}$$

and thus

$$\frac{t}{s} + 1 = \frac{[(\alpha_1+2)w - [\alpha_1+2-\lambda]](\alpha_1+1)v - (\alpha_1+1)^2v^2}{(\alpha_1+1)v - [2\alpha_1 u + 1]} - [[2\alpha_1 u + 1] - (\alpha_1+1)v].$$

Hence $\psi \in \Psi[\Omega, q]$, and by Theorem 1.1, we have

$$g(z) \prec q(z) \text{ or } \frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'} \prec q(z).$$

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(U)$ for some conformal mapping $h(z)$ of U on to Ω, the class $\Phi_{n,1}[h(U), q]$ is written as $\Phi_{n,1}[h, q]$. □

The following result is an immediate consequence of Theorem 2.9.

Theorem 2.10. *Let* $\phi \in \Phi_{n,1}[h(U), q]$ *with* $q(0) = 1$, *if* $f(z) \in \mathcal{A}$ *satisfies the following inclusion relationship* $\phi\left(\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+3]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}; z\right) \prec h(z)$, *then* $\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'} \prec q(z)$.

In the particular case $q(z) = 1 + Mz, M > 0$ and in view of Definition 2.1, the class of admissible function $\Phi_{n,1}[\Omega, q]$, denoted by $\Phi_{n,1}[\Omega, M]$ is described below.

Definition 2.4. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{n,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that:

$$\phi\Bigg(1 + Me^{i\theta}, 1 + \frac{Me^{i\theta}[k+\alpha_1(1+Me^{i\theta})]}{(\alpha_1+1)(1+Me^{i\theta})}, 1 + \frac{[\alpha_1+1-\lambda](1+Me^{i\theta})+[\alpha_1(1+Me^{i\theta})+k]Me^{i\theta}}{(\alpha_1+2)(1+Me^{i\theta})}$$
$$+\frac{(e^{-i\theta}+M)[[\alpha_1\left(1+Me^{i\theta}\right)+1]kM+Le^{-i\theta}]-k^2M^2}{(\alpha_1+2)(e^{-i\theta}+M)[(\alpha_1+1)e^{-i\theta}+\alpha_1 M^2e^{i\theta}+[1+2\alpha_1+k]M]}; z\Bigg) \notin \Omega, \quad (2.19)$$

whenever $z \in U$, $k \geq nM$, $Re(Le^{-i\theta}) \geq (n-1)k$ and $\theta \in R$.

Corollary 2.11. *Let* $\phi \in \Phi_{n,1}[\Omega, M]$. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\phi\left(\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+3]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}; z\right) \in \Omega,$$

then $\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'} \prec 1 + Mz \ (z \in U)$.

In the special case $\Omega = q(U) = \{w : |w-1| < M\}$, the class $\Phi_{n,1}[\Omega, M]$ is defined by $\Phi_{n,1}[M]$ and Corollary 2.11 takes follows for:

Corollary 2.12. *Let* $\phi \in \Phi_{n,1}[M]$. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left|\phi\left(\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}, \frac{(D_z^{-\lambda}H_m^l[\alpha_1+3]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}; z\right) - 1\right| < M,$$

then $\left|\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'} - 1\right| < M$.

Corollary 2.13. *If* $M > 0$ *and* $f(z) \in \mathcal{A}$ *satisfies*

$$\left|\frac{(D_z^{-\lambda}H_m^l[\alpha_1+2]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'} - \frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'}\right| < \frac{M^2}{(\alpha_1+1)(1+M)},$$

then $\left|\frac{(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z))'}{(D_z^{-\lambda}H_m^l[\alpha_1]f(z))'} - 1\right| < M$.

Proof. This follows from Corollary 2.11 by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(U)$ where $h(z) = \frac{M^2}{(\alpha_1+1)(1+M)} z \ (M > 0)$. To use the Corollary 2.11, we need to show that $\phi \in \Phi_{n,1}[M]$, that is the admissibility condition (2.19) is satisfied. This follows since

$$|\phi(u, v, w; z)| = \left|1 + \frac{Me^{i\theta}[k + \alpha_1(1 + Me^{i\theta})]}{(\alpha_1+1)(1+Me^{i\theta})} - 1 - Me^{i\theta}\right|$$
$$= \frac{M}{(\alpha_1+1)}\left|\frac{k-(1+Me^{i\theta})}{(1+Me^{i\theta})}\right| \geq \frac{M}{(\alpha_1+1)}\left|\frac{k-(1+M)}{(1+M)}\right| \geq \frac{M^2}{(\alpha_1+1)(1+M)},$$

whenever $z \in U, k \geq nM, k \neq 1 + M$ and $\theta \in R$. Hence the result is easily deduced from Corollary 2.11. □

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Received 07 March 2013

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