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On D-Compact Groups

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Abstract

In this paper we introduce a new notions in theory of groups for examples D-cover semigroup , *D-compact semigroup* , *D-compact c. semigroup, D-compact locally semigroup, D-cover group, D-compact group, D-compact c. group, D-compact locally group, D-compact strong locally group* , *weak D-compact group* and *weak D-compact c. group*. We mention advance various examples and we introduce some results about our work , for examples ;

1- *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group* .

2- *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group* .

3- *D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact group* .

4- *D-compact strong locally group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group* .

5- *D-compact strong locally group* $\underset{\not\subset}{\Rightarrow}$ *D-compact strong locally semigroup* .

6- *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact semigroup* .

7- Any finite group of order a nonprime number , is a *D-compact locally group* .

8- If $(G,*)$ is a finite group then
D-compact group $\Leftrightarrow$ *D-compact c. group* $\Leftrightarrow$ *weak D-compact group*
$\Leftrightarrow$ *weak D- compact c. group* $\Leftrightarrow$ *D-compact locally group* .

9- Any finite group of order a prime number, is not *D-compact* group .

10- Any commutative simple group is not *D-compact* group .

11- Any cyclic group is not *D-compact* group .

12- If f: $(G,*) \cong (\overline{G},\overline{*})$ is an isomorphism , then $(G,*)$ is a D-*compact group* $\Leftrightarrow$ $(\overline{G},\overline{*})$ is a D-*compact group*.

13- The product of any finite collection of a *D-compact groups* is a *D-compact group*.

Keywords and Phrases: Semigroups , groups , normal subgroup , cover , finite cover , countable cover, direct product , isomorphism .
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1. Introduction and Preliminaries

In [1] , Cauchy contributed in the first attempts at classification on the theory of groups , Galois introduced into the theory the exceedingly important idea of a (normal) subgroup , and thereby contributed greatly, if indirectly, to its subsequent development. Many additions were made, mainly by French mathematicians, during the middle part of the [nineteenth] century. The first connected exposition of the theory was given in the third edition of M. Serret's "Course d'Algebra Suprerieure," which was published in 1866. In recent years, the theory has advanced continuously . There are many types of groups , like , cyclic group , sylow group , Lie group , . . . etc, [3] . This work is inspired from some concepts like covering and compaction in topology and therefore we introduced simply a new concept to the theory of groups which is called a D-compact group .

This work contains two sections ; there are two parts in section1 ,the first part we introduce definitions about our work , and the second part we mention advance various examples . In second section , we introduce some results about our work , for examples ;

1. *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group .*
2. *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group .*
3. *D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact group .*
4. *D-compact strong locally group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group .*
5. *D-compact strong locally group* $\underset{\not\subset}{\Rightarrow}$ *D-compact strong locally semigroup .*
6. *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact semigroup .*
7. Any finite group of order a nonprime number , is a *D-compact locally group .*
8. If (G,*) is a finite group then
   D-compact group $\Leftrightarrow$ *D-compact c. group* $\Leftrightarrow$ *weak D-compact group*
   $\Leftrightarrow$ *weak D- compact c. group* $\Leftrightarrow$ *D-compact locally group .*
9. Any finite group of order a prime number, is not *D-compact group*.
10. Any commutative simple group is not *D-compact group*.
11. Any cyclic group is not *D-compact group*.
12. If *f*: (G,*) $\cong(\overline{G},\overline{*})$ is an isomorphism , then (G,*) is a *D-compact group* $\Leftrightarrow$ $(\overline{G},\overline{*})$ is a *D-compact group*.
13. The product of any finite collection of a *D-compact groups* is a *D-compact group*.

2. Definitions and Examples

[1]**Definition 1 .** Let (G,*) be a semigroup , and

$D=\{(G_i,*);G_i\subset G,(G_i,*)\,\text{is a semigroup},\forall i\in I\}$ be a family of semigroups (G_i,*) indexed by I (I is a finite or an infinite set) , we say that D is a *D-cover semigroup* of (G,*) if $G=\bigcup_{i\in I} G_i$.

Definition 2 . Let (G,*) be a semigroup , we say that (G,*) is a *weak D-compact semigroup* if there is a finite *D-cover semigroup* of (G,*).

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1 Recall that a family $\{G_i\}_{i\in I}$ of subset of the set G is a covering of G if $G=\bigcup_{i\in I} G_i$, [4] .

Definition 3 . Let (G,*) be a semigroup , we say that (G,*) is *D-compact semigroup* if for every *D-cover semigroup* of (G,*) there exists a finite *sub-D-cover semigroup* of (G,*).

Definition 4 . Let (G,*) be a semigroup , we say that (G,*) is *weak D-compact c. semigroup* if there is a countable *D-cover semigroup* of (G,*).

Definition 5 . Let (G,*) be a semigroup , we say that (G,*) is *D-compact c. semigroup* if for every *D-cover semigroup* of (G,*) there exists a countable *sub-D-cover semigroup* of (G,*).

Definition 6 . Let (G,*) be a semigroup , we say that (G,*) is a *D-compact locally semigroup* if for every element x of G there is a subsemigroup of G include x .

[2]**Definition 7 .** Let (G,*) be a semigroup with unite element , (which is called monoid) , we say that (G,*) is a *D-compact strong locally semigroup* if for every element x of G (except the unite element) there is a unique subsemigroup of G include x .

Definition 8 . Let (G,*) be a semigroup , the subsemigroup (H,*) of the semigroup (G,*) is called a *D-compact subsemigroup* (*weak D-compact subsemigroup , weak D-compact c. subsemigroup , D-compact c. subsemigroup , D-compact locally subsemigroup , D-compact strong locally subsemigroup*) , if (H,*) is a *D-compact semigroup* (*weak D-compact semigroup, weak D-compact c. semigroup , D-compact c. semigroup, D-compact locally semigroup, D-compact strong locally semigroup*) , respectively.

Definition 9 . Let (G,*) be a group , and $D=\{(G,*);G_i\subset G,(G_i,*)\text{ is a subgroup of }(G,*),\forall i\in I\}$ be a family of proper subgroups (G_i,*) of (G,*) , indexed by I (I is a finite or an infinite set), we say that D is a D-cover group of (G,*) if $G=\bigcup_{i\in I}G_i$.

Definition 10 . Let (G,*) be a group , we say that (G,*) is a *weak D-compact group* if there is a finite *D-cover group* of (G,*).

[3]**Definition 11 .** Let (G,*) be a group , we say that (G,*) is *D-compact group* if for every *D-cover group* of (G,*) there exists a finite *sub-D-cover group* of (G,*).

Definition 12 . Let (G,*) be a group , we say that (G,*) is weak *D-compact c. group* if for there is a countable *D-cover group* of (G,*).

Definition 13 . Let (G,*) be a group , we say that (G,*) is *D-compact c. group* if for every *D-cover group* of (G,*) there exists a countable *sub-D-cover group* of (G,*).

Definition 14 . Let (G,*) be a group , we say that (G,*) is a *D-compact locally group* if for every element x of G there is a subgroup (proper) of G include x .

Definition 15 . Let (G,*) be a group , we say that (G,*) is a D-*compact strong locally group* if for every element x of G (except the unite element) there is a unique subgroup (proper) of G include x.

Definition 16 . Let (G,*) be a group , the subgroup (H,*) of the group (G,*) is called a *D-compact subgroup* (weak *D-compact subgroup , weak D-compact c. subgroup , D-compact c. subgroup , D-compact locally subgroup , D-compact strong locally subgroup*), if (H,*) is a *D-compact group* (*weak D-compact group, weak D-compact c. group , D-compact c. group , D-compact locally group , D-compact strong locally group*) , respectively .

Definition17. [3] Suppose Λ is non-empty set and $(G_\lambda,*_\lambda)$ is a group for each $\lambda\in\Lambda$. Their *product* is $\prod_{\lambda\in\Lambda}G_\lambda$ with multiplication given by $(x\otimes y)=x_\lambda *_\lambda y_\lambda$ for each $x_\lambda,y_\lambda\in G_\lambda$ and $\lambda\in\Lambda$.

Remark1. If $G_\lambda=G,\forall\lambda\in\Lambda$, then we denoted that $G^\Lambda=\prod_{\lambda\in\Lambda}G_\lambda$.

Definition18. [3] Let (G,*) and $(\overline{G},\overline{*})$ are two groups , we say that

1. $f:(G,*)\to(\overline{G},\overline{*})$ is a *homomorphism* if $f(x*y)=f(x)\,\overline{*}\,f(y)\cdot\forall x,y\in G$.
2. $f:(G,*)\to(\overline{G},\overline{*})$ is an *isomorphism* if f is a *bijective homomorphism* .

Definition19. [3] Let (G,*) and $(\overline{G},\overline{*})$ are two groups , we say that (G,*) is an isomorphic to $(\overline{G},\overline{*})$, denoted that $(G,*)\cong(\overline{G},\overline{*})$, if there is an isomorphism $f:(G,*)\to(\overline{G},\overline{*})$.

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2 See [5] .
3 Similar to ideas likes in , [2] .

Now , we introduce some diverse examples to explain the above definitions

Example 1 . Let (S_3,o) be the symmetric group of degree 3 . the all subgroups of (S_3,o) are $\langle(12)\rangle = \{e,(12)\}, \langle(13)\rangle = \{e,(13)\}, \langle(23)\rangle = \{e,(23)\}$ and $\langle(123)\rangle = \{e,(123),(132)\}$.

The group (S_3,o) is a *D-compact group* , which have not *D-compact subgroup*.

We know that $\langle(123)\rangle$ is a normal subgroup of S_3 , and hence the quotient group $S_3/\langle(123)\rangle$ is not a *D-compact group* , since the number of elements in $S_3/\langle(123)\rangle$ is 2 .

Example 2 . Let G={0,1,2,…, n}, defined a binary operator $\prec$ as follows;

$$a \prec b = \begin{cases} \max\{a,b\} & a \neq b \\ 0 & a = b \end{cases}, \forall\, a,b \in G$$. It is easy to show that $(G,\prec)$ is a group .

The group $(G,\prec)$ is a *D-compact group* , which have a *D-compact subgroups* .

Example 3 . The group (Z_3+$_3$) , has no *D-cover semigroup* and hence has no *D-cover group* . The group (Z_3+$_3$) is not *D-compact group* **,** while the group (Z_3+Z_3, $\oplus$) is a *D-compact group* . the subgroup {(0,0),(0,1),(0,2),(0,3),(2,0),(2,1),(2,3),(2,2)} is a *D-compact subgroup* of (Z_3+Z_3, $\oplus$).

Example 4 . The group (Z,+) , has *D-cover semigroup* , since (Z^+,+), (Z^-,+) and ({0},+) are semigroup and Z=Z^- U {0} U Z^+ , which has not *D-cover group* , and hence the group (Z,+) is not *D-compact group* .

Example 5 Let G={0,1,2, …} , defined a binary operator $\prec$ as follows ;

$$a \prec b = \begin{cases} \max\{a,b\} & a \neq b \\ 0 & a = b \end{cases}, \forall\, a,b \in G$$. It is easy to show that $(G,\prec)$ is a group .

The group $(G,\prec)$ is a *D-compact C. group* . The group $(G,\prec)$ is not *D-compact group* , since the family of subgroups {({0,a,b},$\prec$); a,b $\in$ N}is a *D-cover group* of $(G,\prec)$ has no finite *sub-D-cover group* of $(G,\prec)$.

Example 6 . Let G=[0,1] , defined a binary operator on G as follows ;

$$a \prec b = \begin{cases} \max\{a,b\} & a \neq b \\ 0 & a = b \end{cases}, \forall\, a,b \in G$$. It is easy to show that $(G,\prec)$ is a group . let

$$D = \left\{\left(\left[0,\frac{1}{n}\right],\prec\right); \forall_n = 2,3,4,\ldots\right\} \mathrm{U} \left(\left\{\left[\frac{1}{2},1\right] \mathrm{U}\{0\}\right\},\prec\right) ,$$

It is clear the set D is a *D-cover group* of ([0,1],*) , since

$$[0,1] = \left\{\left[\frac{1}{2},1\right] \mathrm{U}\{0\}\right\} \mathrm{U}\left[0,\frac{1}{2}\right] \mathrm{U}\left[0,\frac{1}{3}\right] \mathrm{U}\left[0,\frac{1}{4}\right] \ldots \mathrm{U}\left[0,\frac{1}{n}\right] \mathrm{U} \ldots$$

There is a finite *sub D-cover group* of ([0,1], $\prec$) . And ([0,1], $\prec$) is a *weak D-compact group* . But ([0,1], $\prec$) is not *D-compact group* (not *D-compact c. group*) because there is a *D-cover group* of ([0,1], $\prec$) which is {([r,1]U{0},$\prec$);0 <r $\leq$ 1} has no finite (countable) *sub D-cover group* .

Let $A = \left\{\frac{1}{n}; n = 1,2,\ldots,10\right\} \mathrm{U}\{0\}$, it is clear that (A, $\prec$) is a *D-compact group* . Let $H = \left\{0,\frac{1}{2}\right\}$, $S = \left\{0,\frac{1}{3}\right\}$ and $H\mathrm{U}S = \left\{0,\frac{1}{2},\frac{1}{3}\right\}$ it is clear that (H U S, $\prec$) is a *D-compact subgroup* of (A, $\prec$) , while (H, $\prec$) and (S, $\prec$) are not *D-compact subgroup* of (A, $\prec$).

Example 7 . The semigroup $(Z,\succ)$, where $\succ$ is a binary operator defined as follows ; $a \succ b = \min\{a,b\}, \forall a,b \in Z$, is a *D-compact c. semigroup* , which is not *D-compact c. group* .

Example 8 . Let X={0}U R^+ , defined a binary operator $\prec$ as follows ;

$a \prec b = \begin{cases} \max\{a,b\} & a \neq b \\ 0 & a = b \end{cases}, \forall\, a,b \in X$. It is easy to show that (X, $\prec$) is a group .

The group (X, $\prec$) is a *D-compact locally group* .

The group (X, $\prec$) is not *D-compact c. group* , since the family of subgroups {({0,*a*,*b*}, (X, $\prec$); *a*, *b* $\in$ R$^+$}is a *D-cover group* of (X, $\prec$) has no countable *sub-D-cover group* of (X, $\prec$).

Let *H*=0,1,2, … } and *S*=[0,10] , it is easy to show that ($H \cap S$, $\prec$) which is a *D-compact subgroup* , while (H, $\prec$) and (S, $\prec$) are not *D-compact subgroups* .

Let *D*={({0} U [*n*, *n*+1), $\prec$); *n*=0,1,2, …}, it is easy to show that *D* is a *D-cover group* and hence (X, $\prec$) is a *weak D-compact c. group*.

Example 9 . Let $a,b \in Z$ such that $b \neq 0$. We know that $\left(\langle\frac{a}{b}\rangle,.\right)$ is a cyclic subgroups of $(Q \setminus \{0\},.)$, where $\langle\frac{a}{b}\rangle = \left\{\left(\frac{a}{b}\right)^n ; n \in Z\right.$, (the rational numbers Q is a countable set) .

Let $D = \{ \left(\langle\frac{a}{b}\rangle,.\right); a,b \in Z, b \neq 0$ it is clear that *D* is a *D-cover group* of (Q\{0}, .). And hence the group (Q\{0}, .) is a *D-compact c. group* . Which is not *weak D-compact group* , since the family of subgroups $\{ \left(\langle\frac{a}{b}\rangle,.\right) ; a,b \in Z, b \neq 0 \}$has no finite *sub-D-cover group* of (Q\{0}, .).

Example 10 . The semigroup (Z, $\succ$) , where $\succ$ is a binary operator defined as follows ; $a \succ b = \min\{a,b\}, \forall\, a,b \in Z$, is a *D-compact c. semigroup* , which is not *D-compact semigroup*, since the family of semigoups $\{ (\{0, z\}, >), z \in Z$ is a *D-cover semigroup* of (Z, $\succ$), which has no finite *sub-D-cover semigroup* of (Z, $\succ$).

Example 11 . The semigroup (Z ,.) , is a *weak D-compact semigroup* . Since (2Z ,.) and (2Z + 1 ,.) are semigroup and $Z = 2Z \bigcup 2Z + 1$. But The semigroup (Z ,.) , is not a *D-compact semigroup* . Since the family of semigoups $\{(nZ,.) ; n = 2, 3, \dots\} \cup (\{1, -1\},.)$ is a *D-cover semigroup* of (Z ,.) which has no finite *sub-D-cover semigroup* of (Z ,.).

Example 12 . Let (R, $\succ$) , where $\succ$ is a binary operator defined as follows ;

$a \succ b = \min\{a,b\}, \forall\, a,b \in R$, it is easy to show that (R, $\succ$) is a semigroup .

The semigroup (R, $\succ$) is a *D-compact locally semigroup* , which is not *D-compact c. semigroup* since the family of semigoups $\{(\{a,b\},\succ), a,b \in R\}$ is a *D-cover semigroup* of (R, $\succ$) which has no countable *sub-D-cover semigroup* of (R, $\succ$).

Example 13 . The semigroup (Q, .) is a *D-compact c. semigroup* , which is not *weak D-compact semigroup* .

Example 14 . Let (R, $\succ$) where $\succ$ is a binary operator defined as follows ;

$a \succ b = \min\{a,b\}, \forall\, a,b \in R$, it is easy to show that (R, $\succ$) is a semigroup .

The semigroup (R, $\succ$) is a *weak D-compact semigroup* , since (R$^+$, $\succ$) , (R$^-$, $\succ$) and ({0},$\succ$) are semigroup and R=R$^-$ $\bigcup$ {0} R$^+$.

And we know that (R, $\succ$) is not a *D-compact c. semigroup* (Example12).

Example 15 . Let $(Z_2 \times Z_2, \oplus)$, where $\oplus$ define as follows ;

The group $(Z_2 \times Z_2, \oplus)$ is a *D-compact strong locally group* , since there are only three subgroups of $(Z_2 \times Z_2, \oplus)$ which are $(D_1,\oplus)$, $(D_2,\oplus)$ and $(D_3,\oplus)$,(except the trivial subgroups) where D_1 = {(0,0),(1,1)} , D_2 = {(0,0), (1,0)} , D_3 = {(0,0), (1,1)}.

Example 16 . The semigroup (Z_3,$._3$) is a *D-compact strong locally semigroup* , since the only subsemigroup of (Z_3,$._3$) are ({0,1},$._3$) and ({1,2},$._3$) , (except the trivial subsemigroups) .

Example 17 . [3] The symmetric group (S_4, o) of degree 4, is a *D-compact group*.

Let $H=\left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix},\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\right\}$

and $S=\left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix},\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}\right\}$

which are a *D-compact subgroups* of the *D-compact group* (S_4, o) .

Example 18 . The group $(\mathrm{R}^+, .)$ is not *weak D-compact c. group* which is a *D-compact locally group* . Since $\forall\, x\in \mathrm{R}^+ \Rightarrow (\langle x\rangle, .)$ is a subgroup of $(\mathrm{R}^+, .)$.

Example 19 . The symmetric group (S_n, o) , of degree $n \geq 3$. is a *D-compact group* **.**

3. Main Results

The prove of all the following lemmas are direct from definitions .

Lemma 1. *D-cover group* $\underset{\not\subset}{\Rightarrow}$ *D-cover semigroup* . ($\not\Leftarrow$ See Example 4) .

Lemma 2 . *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact semigroup* .

Lemma 3 . *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact c. group* . ($\not\Leftarrow$See Example 5) .

Lemma 4 . *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact group* . ($\not\Leftarrow$ See Example 6) .

Lemma 5 . *D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group* . ($\not\Leftarrow$ See Example 5 and Example 6) .

Lemma 6 . *D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *D-compact c. semigroup* . ($\not\Leftarrow$ See Example 7) .

Lemma 7 . *D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group* . ($\not\Leftarrow$ See Example 8) .

Lemma 8 . *D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact group* . ($\not\Rightarrow$ See Example 9) . ($\not\Leftarrow$ See Example 6) .

Lemma 9 . *weak D-compact group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group* . ($\not\Leftarrow$ See Example 9) .

Lemma 10 . *D-compact semigroup* $\underset{\not\subset}{\Rightarrow}$ *D-compact c. semigroup* . ($\not\Leftarrow$ See Example 10) .

Lemma 11 . *D-compact semigroup* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact semigroup* . ($\not\Leftarrow$ See Example 11).

Lemma 12 . *D-compact semigroup* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally semigroup* .($\not\Leftarrow$ See Example 11) .

Lemma 13 . *D-compact c. semigroup* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally semigroup* . ($\not\Leftarrow$ See Example12).

Lemma 14 . *D-compact c. semigroup* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact semigroup* . ($\not\Rightarrow$ See Example 13) . ($\not\Leftarrow$ See Example 14) .

Lemma 15 . *weak D-compact semigroup* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally semigroup* . ($\not\Leftarrow$ See Example 13) .

Lemma 16 . *D-compact locally group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally semigroup* .

Lemma 17 . *weak D-compact group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact semigroup* .

Lemma 18 . *D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact c. group* . ($\not\Leftarrow$ See Example 8) .

Lemma 19 . *weak D-compact group* $\underset{\not\subset}{\Rightarrow}$ *weak D-compact c. group* . ($\not\Leftarrow$ See Example 9) .

Lemma 20 . *weak D-compact c. group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group* . ($\not\Leftarrow$ See Example 8) .

Lemma 21 . *D-compact strong locally group* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally group* . ($\not\Leftarrow$ See Example 9) .

Lemma 22 . *D-compact strong locally semigroup* $\underset{\not\subset}{\Rightarrow}$ *D-compact locally semigroup*. ($\not\Leftarrow$ See Example 12) .

Lemma 23 . *D-compact strong locally group* $\underset{\not\subset}{\Rightarrow}$ *D-compact strong locally semigroup*. ($\not\Leftarrow$ See Example 16).

Theorem 1 . Any finite group of order a nonprime number , is a *D-compact locally group* .

Proof .

Let $(G, *)$ is a finite group , for every element x of G the subgroup $(\langle x\rangle,*)$ of $(G, *)$ include x , it is clear that $G \neq \langle x\rangle$, since G is not cyclic group , and therefore $(G, *)$is a *D-compact locally group* . □

The following theorem is direct

Theorem 2 . If $(G, *)$ is a finite group . Then the following are equivalent ;

1) $(G, *)$ *D-compact group,*
2) $(G, *)$ *D-compact c. group,*
3) $(G, *)$ *weak D-compact group,*
4) $(G, *)$ *weak D-compact c. group,*
5) $(G, *)$ *D-compact locally group.*

Theorem 3 . Any finite group has a prime order is not *D-compact locally group* .

Proof .

Let $(G, *)$ is a group with $|G|=p$, p prime number , by "Lagrange theorem" the order of every subgroup of G divides p , but p is prime , so there is no proper subgroup of G except the unit element and hence $(G, *)$ is not *D-compact locally group* . □

We can prove directly from Theorem2 and Theorem3 , respectively . the following corollaries ;

Corollary 1 . Any finite group has prime order is not *D-compact group* .

Corollary 2 . Any finite group has prime order is not *D-compact c. group* .

Corollary 3 . Any finite group has prime order is not *weak D-compact group* .

Corollary 4 . Any finite group has prime order is not *weak D-compact c. group* .

The following corollary is direct from Lemma21 and Theorem3 ;

Corollary 5 . Any finite group has prime order is not *D-compact strong locally group* .

Theorem 4 . If (*G*, *) is a commutative group . Then (*G*, *)is a *D-compact group* $\underset{\not\subset}{\Rightarrow}$ (*G*, *) is not simple group .

Proof .

($\Rightarrow$) If (*G*, *) is a *D-compact group* then there is *D-cover group* and hence there is a subgroup of (*G*, *) $\Rightarrow$ (*G*, *) is not simple group.

($\not\Leftarrow$) See Example 4 . □

Theorem 5 . Any commutative simple group is not *weak D-compact group*.

Proof .

If (*G*, *) is a commutative group then any subgroup of (*G*, *) become necessary a normal subgroup so (*G*, *) have no subgroup and hence have no *D-cover group* . □

The following corollary is direct from Lemma4 and Theorem5 ;

Corollary 6 . Any commutative simple group is not *D-compact group* .

We can prove directly the following theorem ,

Theorem 6 . Any commutative simple group is not *D-compact c. group* .

Theorem 7 . Any cyclic group is not *D-compact locally group* .

Proof .

Assume (*G*, *) is a cyclic group which is a *D-compact locally group* so for every element *x* of *G* there is a subgroup of *G* include *x* , but *G* is a cyclic so there is an element say *g* such that $\langle g\rangle = G$ (*G* generated by *g*) and hence any subgroup contains *g* must be equal to *G*, that is there is no proper subgroup contains *g* . □

We can prove directly from Lemma 5, Lemma 7, Lemma 9 , Lemma 20 and Lemma 21 , respectively, the following corollaries ,

Corollary 7 . Any cyclic group is not *D-compact group* .

Corollary 8 . Any cyclic group is not *D-compact c. group* .

Corollary 9 . Any cyclic group is not *weak D-compact group* .

Corollary 10 . Any cyclic group is not *weak D-compact c. group* .

Corollary 11 . Any cyclic group is not *D-compact strong locally group* .

Theorem 8 . If (*H*,*) and (*S*,*) are two *subgroups* of *D-compact group* (*G*,*) . Then

1. (*H*,*) and (*S*,*) are two *D-compact subgroups* $\underset{\not\subset}{\Rightarrow}$ ($H \cap S$, *) is a *D-compact subgroup* .
2. (*H*,*) and (*S*,*) are two *D-compact subgroups* $\underset{\not\subset}{\Rightarrow}$ ($H \cup S$, *) is a *D-compact subgroup*.

Proof . 1- ($\not\Rightarrow$) See Example 17 **.** ($\not\Leftarrow$) See Example 8**.**

2- ($\not\Rightarrow$) See Example 17 **.** ($\not\Leftarrow$) See Example 6**.**

Theorem 9 . If $(G,*) \cong (\overline{G},\overline{*})$. Then

(G,)* is a *D-compact group* $\Leftrightarrow$ $(\overline{G},\overline{*})$ is a *D-compact group*.

Proof .

($\Rightarrow$) Let $(\overline{G}_l,\overline{*}_l)$ be any *D-cover group* of $(\overline{G},\overline{*}) \Rightarrow \overline{G} = \bigcup_{i\in I}\overline{G}_l$, but f is an isomorphism $\Rightarrow f(G) = \overline{G} = \bigcup_{i\in I}\overline{G}_l \Rightarrow G = f^{-1}(\bigcup_{i\in I}\overline{G}_l) = \bigcup_{i\in I} f^{-1}(\overline{G}_l)$, and $f^{-1}(\overline{G}_l)$ is a group $\forall i \in I$, but (*G*,*) is a *D-compact group* so there is a finite set *J* such that

[4] $G=\bigcup_{j\in J} f^{-1}(\overline{G}_j)=f^{-1}(\bigcup_{j\in J}\overline{G}_J)\Rightarrow \overline{G}=f(G)=f\left(f^{-1}(\bigcup_{j\in J}\overline{G}_J)\right)=\bigcup_{j\in J}\overline{G}_J$ and $(\overline{G}_J,\overline{*}_J)$ is a group $\forall j\in J\Rightarrow(\overline{G},\overline{*})$ is a *D-compact group* .

$(\Leftarrow)$ Let $(G_i,*_i)$ be any *D-cover group* of $(G,*)\Rightarrow G=\bigcup_{i\in I}G_i$, but f is an isomorphism $\Rightarrow \overline{G}=f(G)=f(\bigcup_{i\in I}G_i)=\bigcup_{i\in I}f(G_i)$, and is a group $\forall i\in I$, but $(\overline{G},\overline{*})$ is a *D-compact group* so there is a finite set *J* such that

[5] $\overline{G}=\bigcup_{j\in J} f(G_j)=f(\bigcup_{j\in J}G_j)\Rightarrow G=f^{-1}(\overline{G})=f^{-1}\left(f(\bigcup_{j\in J}G_j\right)=\bigcup_{j\in J}G_j$ and $(\overline{G}_j,\overline{*}_j)$ is a group $\forall j\in J\Rightarrow(G,*)$ is a *D-compact group* . □

Corollary 12 . If $f:(G,*)\rightarrow(\overline{G},\overline{*})$ is an isomorphism and $(H,*)$ is a *D-compact subgroup* of $(G,*)$. Then *f*(*H*) is a *D-compact subgroup* of $(\overline{G},\overline{*})$.

Corollary 13 . If $f:(G,*)\rightarrow(\overline{G},\overline{*})$ is an isomorphism and $(S,*)$ is a *D-compact subgroup* of $(\overline{G},\overline{*})$. Then f^{1} (*S*) is a *D-compact subgroup* of $(G,*)$.

Theorem 10 . If (*A*,* (*A*,*) is a group and (G, *) is a *D-compact group* . Then $(A\times G,\otimes)$ is a *D-compact group* .

Where $(a_1,g_1)\otimes(a_2,g_2)=(a_1*a_2,\,g_1*g_2)$, $\forall\,(a_1,g_1),(a_2,g_2)\in A\times G$.

Proof .

Let $\{(A\times G_i,\otimes);G_i\subset G,(A\times G_i,\otimes)$ is a subgroup of $(A\times G,\otimes)$, $\forall i\in I$ be any *D-cover group* of $(A\times G,\otimes)$, it is clear that (G_i,*) is a subgroups of (*G*,*) , such that $A\times G=\bigcup_{i\in I}A\times G_i=A\times(\bigcup_{i\in I}G_i)\Rightarrow G=\bigcup_{i\in I}G_i$, but (*G*,*) is a *D-compact group* , so there is a finite set *J* such that $G=\bigcup_{i\in I}G_j$, and hence

$A\times G=A\times(\bigcup_{i\in J}G_j)\bigcup_{i\in J}A\times G_i\ \Rightarrow\ (A\times G,\otimes)$ is a *D-compact group* . □

Theorem 11 . If $(G,*)$ and $(\overline{G},\overline{*})$ are *D-compact strong locally groups* . Then $(G\times\overline{G},\otimes)$ is a *D-compact strong locally group* .

Proof .

Let $(x,y)\in G\times\overline{G}\Rightarrow x\in G$ and $y\in\overline{G}$, but $(G,*)$ and $(\overline{G},\overline{*})$ are *D-compact strong locally groups* $\Rightarrow\exists!G_x$ and $\exists!\overline{G}_y$ subgroups of *G* and $\overline{G}$, respectively, such that $x\in G_x$ and $y\in\overline{G}_y\Rightarrow(x,y)\in G_x\times\overline{G}_y$ and $G_x\times\overline{G}_y$ is a unique subgroups of $G\times\overline{G}\Rightarrow(G\times\overline{G},\otimes)$ is a *D-compact strong locally group* . □

It is easy to prove the following theorem by the same way of prove Theorem11 ;

Theorem 12 . If $(G,*)$ and $(\overline{G},\overline{*})$ are *D-compact locally groups* . Then $(G\times\overline{G},\otimes)$ is a *D-compact locally group* .

Theorem 13 . If $(G,*)$ and $(\overline{G},\overline{*})$ are *D-compact groups* $\underset{\not\subset}{\Rightarrow}(G\times\overline{G},\otimes)$ is a *D-compact group* .

Proof .

$(\Rightarrow)$ Let (*G*, *) and $(\overline{G},\overline{*})$ are *D-compact groups* $\Rightarrow$ there exists a *D-cover group* of (*G*,*) say $G\{G_a\}_{a\in A}$ and a *D-cover group* of $(\overline{G},\overline{*})$ say $\{\overline{G}_b\}_{b\in B}$

$\Rightarrow G\times\overline{G}=(\bigcup_{a\in A}G_a)\times(\bigcup_{b\in B}\overline{G}_b)=\bigcup_{a\in A,b\in B}(G_a\times\overline{G}_b)$

$\Rightarrow\{G_a\times\overline{G}_b\}_{a\in A,b\in B}$ is a *D-cover group* of $(G\times\overline{G},\otimes)$.

Let $\{W_i\}_{i\in I}$ be any *D-cover group* of $(G\times\overline{G},\otimes)\Rightarrow G\times\overline{G}=\bigcup_{i\in I}W_i$, such that $w_i=u_i\times v_i$, where $\{u_i\}_{i\in I}$ are subgroups of (*G*, *) and $\{V_i\}_{i\in I}$ are subgroups of $(\overline{G},\overline{*})$. But (*G*, *) is a *D-compact group* , so

[4] See [4]
[5] See [4]

there is a *D-cover group* of $(G, *)$ contains $\{u_i\}_{i\in I}$ which have a finite *sub-D-cover group* (*i.e.* there is a finite set J) such that $G=\bigcup_{j\in J} u_j$, let $u_{j_1} \in \{u_j\}_{j\in J} \Rightarrow \{u_{j_1} \times v_i\}_{i\in I}$ is a *D-cover group* of the *D-compact group* $(u_{j_1} \times \overline{G}, \otimes)$ (from Theorem 10 since $(u_{j_1}, *)$ is a group and $(\overline{G}, \overline{*})$ is a *D-compact group*) , so there is a finite set S such that

$$u_{j_1} \times \overline{G} = \bigcup_{s\in S}(u_{j_1} \times v_s) = u_{j_1} \times (\bigcup_{s\in S} v_s)$$

$$\Rightarrow \bigcup_{j\in J}(u_j \times (\bigcup_{s\in S} v_s)) = (\bigcup_{j\in J} u_j) \times (\bigcup_{s\in S} v_s) = G \times \overline{G}$$

$\Rightarrow G \times \overline{G} = (\bigcup_{j\in J} u_j) \times (\bigcup_{s\in S} v_s) = \bigcup_{j\in J, s\in S}(u_j \times v_s)$, where $u_j \times v_s$ are subgroups of $G \times \overline{G}$. And therefore $G \times \overline{G}$ is a *D-compact group* .

($\not\Leftarrow$) See Example 3 **.** □

It is easy to prove the following Corollary ;

Corollary 14 . If $(G,*)$ is a *D-compact group* (*D-compact strong locally group* , *D-compact locally group* , *weak D-compact group* , *weak D-compact c. group*) . Then $(G^2, \otimes)$ is a *D-compact group* (*D-compact strong locally group* , *D-compact locally group* , *weak D-compact group* , *weak D-compact c. group*) , respectively .

By induction we can prove the following theorems ,

Theorem 14 . If $(G,*)$ is a *D-compact group* (*D-compact strong locally group* , *D-compact locally group* , *weak D-compact group* , *weak D-compact c. group*) . Then $(G^n, \otimes)$ is a *D-compact group* (*D-compact strong locally group* , *D-compact locally group* , *weak D-compact group* , *weak D-compact c. group*) , respectively , for each $n \in \mathbb{N}$.

Theorem 15 . The product of any finite collection of *D-compact groups* (*D-compact strong locally groups* , *D-compact locally groups* , *weak D-compact groups* , *weak D-compact c. groups*) , is a *D-compact group* (*D-compact strong locally group* , *D-compact locally group* , *weak D-compact group* , *weak D-compact c. groups*) .

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