

Differential Subordination Theorems of Analytic Functions and Some Applications

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Abstract

In the present paper , we study some classes of analytic functions , we obtain a number of sufficient conditions for normalized analytic functions in the unit disk . Also we give some applications of the first – order differential subordination for generalized Briot-Bouquet differential subordination .

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1. Introduction and preliminaries

Let $L(\lambda)$ denote the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^{n-\lambda} \quad (0 \leq \lambda < 1), \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ and satisfying $f(0) = f'(0) - 1 = 0$.

Also let $T(\lambda)$ denote the class of all functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^{n-\lambda} \quad (0 \leq \lambda < 1, a_n \geq 0), \quad (1.2)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ and satisfying $f(0) = f'(0) - 1 = 0$.

A function $f \in L(\lambda)$ is said to be starlike of order β , if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \quad (z \in U, 0 \leq \beta < 1). \quad (1.3)$$

Denote this class by $S^*(\beta)$.

A function $f \in L(\lambda)$ is said to be convex of order β , if

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \quad (z \in U, 0 \leq \beta < 1). \tag{1.4}$$

Denote this class by $C(\beta)$.

For two functions f and F analytic in U , we say f is subordinate to F in U , denote by $f(z) \prec F(z)$ or $f \prec F$, if there exists a schwarz function $w(z)$ analytic in U , with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z)), (z \in U)$.

In particular, if the function F is univalent in U , then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let h be univalent in U . Assume that p, ϕ are analytic and univalent in U if p satisfies the differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.5}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply dominant if $p \prec q$ for all p satisfying (1.5). A dominant \tilde{q} that satisfies $\tilde{q} \prec q$ for all dominants q of (1.5) is said to be the best dominant of (1.5).

Let $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \dots$. The Gaussian hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(z) = {}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \dots (\lambda + n - 1) & (n \in \mathbb{N}). \end{cases}$$

Definition 1.1. [1] Let $0 \leq \lambda < 1$ and $\mu, v \in \mathbb{R}$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda, \mu, v}$ of a function f is defined by:

$$J_{0,z}^{\lambda, \mu, v} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-\varepsilon)^{-\lambda} f(\varepsilon) \cdot {}_2F_1\left(\mu-\lambda, -v; 1-\lambda; 1-\frac{\varepsilon}{z}\right) d\varepsilon \right\}, & (0 \leq \lambda < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\lambda-n, \mu, v} f(z), & (n \leq \lambda < n+1, n \in \mathbb{N}), \end{cases} \tag{1.7}$$

where the function f is analytic in a simply-connected region of the z -plane containing the origin, with the order

$$f(z) = O(|z|^\epsilon), (z \rightarrow 0),$$

for $\epsilon > \max\{0, \mu - v\} - 1$, and the multiplicity of $(z - \varepsilon)^{-\lambda}$ is removed by requiring $\log(z - \varepsilon)$ to be real, when $(z - \varepsilon) > 0$.

Definition 1.2. [1] The fractional derivative of order λ , ($0 \leq \lambda < 1$) of a function f is defined by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\varepsilon)}{(z-\varepsilon)^\lambda} d\varepsilon, \tag{1.8}$$

where f it is chosen as in Definition 1.1, and the multiplicity of $(z - \varepsilon)^{-\lambda}$ is removed by requiring $\log(z - \varepsilon)$ to be real, when $(z - \varepsilon) > 0$.

By comparing (1.7) with (1.8), we find

$$J_{0,z}^{\lambda, \lambda, v} f(z) = D_z^\lambda f(z), \quad (0 \leq \lambda < 1). \tag{1.9}$$

In terms of gamma function ,we have

$$J_{0,z}^{\lambda,\mu,v} z^n = \frac{\Gamma(n+1)\Gamma(n-\mu+v+1)}{\Gamma(n-\mu+1)\Gamma(n-\lambda+v+1)} z^{n-\mu}, \tag{1.10}$$

$$(0 \leq \lambda < 1, \mu, v \in \mathbb{R} \text{ and } n > \max \{0, \mu - v\} - 1).$$

Lemma 1.1. [5] Let q be univalent in the unit disk U and θ and ϕ be analytic in domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$.Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that

1. $Q(z)$ is starlike univalent in U , and
2. $Re \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$ for $z \in U$.

If $\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) < q(z)$ and q is the best dominant .

Lemma 1.2. [6] Let q be convex univalent in the unit disk U and ψ and $t \in \mathbb{C}$ with $Re \left\{ 1 + \frac{zq''(z)}{q'(z)} + \psi t \right\} > 0$. If p is analytic in U and $\psi pz + tzp'z < \psi qz + tqz'$, then $p < q$ and q is the best dominant .

2. Main Result

Theorem 2.1. Let the function q be univalent in the unit disk U , $q(z) \neq 0$ and assume that

$$Re \left\{ 1 + \frac{\lambda_2}{t} q(z) + \frac{2\lambda_3}{t} q^2(z) + \frac{3\lambda_4}{t} q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0, \tag{2.1}$$

where $\lambda_i \in \mathbb{C}$, $i = 1,2,3,4$, $t \in \mathbb{C} \setminus \{0\}$.

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U . If $f \in L(\lambda)$ satisfies

$$\begin{aligned} \lambda_1 + \lambda_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + \lambda_3 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{2\delta} + \lambda_4 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{3\delta} + t\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \\ < \lambda_1 + \lambda_2 q(z) + \lambda_3 q^2(z) + \lambda_4 q^3(z) + t \frac{zq'(z)}{q(z)}, \end{aligned} \tag{2.2}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < q(z), \quad z \in U, f(z) \neq -z$$

and q is the best dominant .

Proof . Define the function p by

$$p(z) = \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta, \quad z \in U, f(z) \neq -z. \tag{2.3}$$

Note that

$$\lambda_1 + \lambda_2 p(z) + \lambda_3 p^2(z) + \lambda_4 p^3(z) + t \frac{zp'(z)}{p(z)} = \lambda_1 + \lambda_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + \lambda_3 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{2\delta}$$

$$+ \lambda_4 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{3\delta} + t\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right]. \tag{2.4}$$

From (2.2) and (2.4), we have

$$\lambda_1 + \lambda_2 p(z) + \lambda_3 p^2(z) + \lambda_4 p^3(z) + t \frac{zp'(z)}{p(z)} < \lambda_1 + \lambda_2 q(z) + \lambda_3 q^2(z) + \lambda_4 q^3(z) + t \frac{zq'(z)}{q(z)}. \tag{2.5}$$

By setting

$$\theta(w) = \lambda_1 + \lambda_2 w + \lambda_3 w^2 + \lambda_4 w^3 \text{ and } \phi(w) = \frac{t}{w}, w \neq 0,$$

we see that $\theta(w)$ is analytic in \mathbb{C} , $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = t \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \lambda_1 + \lambda_2 q(z) + \lambda_3 q^2(z) + \lambda_4 q^3(z) + t \frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U ,

$$Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = Re \left\{ 1 + \frac{\lambda_2}{t} q(z) + \frac{2\lambda_3}{t} q^2(z) + \frac{3\lambda_4}{t} q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\}. \tag{2.6}$$

From (2.1) and (2.6), we have

$$Re \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0.$$

Therefore by Lemma 1.1, we get $p(z) < q(z)$. By using (2.3), we obtain the result.

By fixing $\lambda_1 = \lambda_3 = \lambda_4 = 0$ and $\lambda_2 = t = \delta = 1$ in the Theorem 2.1, we obtain the following corollary:

Corollary 2.1. Let the function q be univalent in the unit disk U , $q(z) \neq 0$ and assume that

$$Re \left\{ 1 + q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0.$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U . If $f \in L(\lambda)$ satisfies

$$1 + \frac{zf''(z)}{1 + f'(z)} < q(z) + \frac{zq'(z)}{q(z)},$$

then

$$\frac{z + zf'(z)}{z + f(z)} < q(z),$$

and q is the best dominant.

By taking $q(z) = (1 + Az)/(1 + Bz)$ ($-1 \leq B < A \leq 1$) in the Theorem 2.1, we obtain the following corollary:

Corollary 2.2. Let the function q be convex univalent in the unit disk U and assume that

$$Re \left\{ \frac{\lambda_2}{t} \left[\frac{1 + Az}{1 + Bz} \right] + \frac{2\lambda_3}{t} \left[\frac{1 + Az}{1 + Bz} \right]^2 + \frac{3\lambda_4}{t} \left[\frac{1 + Az}{1 + Bz} \right]^3 + \frac{1 - ABz^2}{(1 + Az)(1 + Bz)} \right\} > 0.$$

If $f \in L(\lambda)$ satisfies

$$\begin{aligned} & \lambda_1 + \lambda_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + \lambda_3 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{2\delta} + \lambda_4 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{3\delta} + t\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \\ & < \lambda_1 + \lambda_2 \left[\frac{1 + Az}{1 + Bz} \right] + \lambda_3 \left[\frac{1 + Az}{1 + Bz} \right]^2 + \lambda_4 \left[\frac{1 + Az}{1 + Bz} \right]^3 + \frac{t(A - B)z}{(1 + Az)(1 + Bz)}, \end{aligned}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1$$

and $q(z) = \frac{1 + Az}{1 + Bz}$ is the best dominant .

By taking $q(z) = e^{\gamma Az}$, $\gamma \neq 0$ in the Theorem 2.1 , we obtain the following corollary :

Corollary 2.3. Let the function q be convex univalent in the unit disk U and assume that

$$Re \left\{ 1 + \frac{\lambda_2}{t} e^{\gamma Az} + \frac{2\lambda_3}{t} e^{2\gamma Az} + \frac{3\lambda_4}{t} e^{3\gamma Az} \right\} > 0 .$$

If $f \in L(\lambda)$ satisfies

$$\begin{aligned} & \lambda_1 + \lambda_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + \lambda_3 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{2\delta} + \lambda_4 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{3\delta} + t\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \\ & < \lambda_1 + \lambda_2 e^{\gamma Az} + \lambda_3 e^{2\gamma Az} + \lambda_4 e^{3\gamma Az} + t\gamma Az \quad , \end{aligned}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < e^{\gamma Az}, \quad \gamma \neq 0$$

and $q(z) = e^{\gamma Az}$ is the best dominant .

By taking $q(z) = \left[\frac{1+z}{1-z} \right]^\gamma$, $\gamma \neq 0$ in the Theorem 2.1 , we obtain the following corollary:

Corollary 2.4. Let the function q be convex univalent in the unit disk U and assume that

$$Re \left\{ \frac{\lambda_2}{t} \left[\frac{1+z}{1-z} \right]^\gamma + \frac{2\lambda_3}{t} \left[\frac{1+z}{1-z} \right]^{2\gamma} + \frac{3\lambda_4}{t} \left[\frac{1+z}{1-z} \right]^{3\gamma} + \frac{z^2 + 1}{(1-z)(1+z)} \right\} > 0 .$$

If $f \in L(\lambda)$ satisfies

$$\begin{aligned} & \lambda_1 + \lambda_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + \lambda_3 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{2\delta} + \lambda_4 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{3\delta} + t\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \\ & < \lambda_1 + \lambda_2 \left[\frac{1+z}{1-z} \right]^\gamma + \lambda_3 \left[\frac{1+z}{1-z} \right]^{2\gamma} + \lambda_4 \left[\frac{1+z}{1-z} \right]^{3\gamma} + \frac{2\gamma tz}{(1-z)(1+z)}, \end{aligned}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < \left[\frac{1+z}{1-z} \right]^\gamma, \quad \gamma \neq 0$$

and $q(z) = \left[\frac{1+z}{1-z} \right]^\gamma$ is the best dominant .

Theorem 2.2. Let the function q be univalent in the unit disk U , $q(z) \neq 0$ and assume that

$$Re \left\{ 1 + \frac{\eta_1 k}{t} + \frac{\eta_2(k+1)}{t} q(z) + \frac{zq''(z)}{q'(z)} + (k-1) \frac{zq'(z)}{q(z)} \right\} > 0, \tag{2.7}$$

where $\eta_1, \eta_2, k \in \mathbb{C}$, $t \in \mathbb{C} \setminus \{0\}$.

Suppose that $z(q(z))^{k-1} q'(z)$ is starlike univalent in U . If $f \in L(\lambda)$ satisfies

$$\begin{aligned} \eta_1 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} + \eta_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta(k+1)} + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \\ < [\eta_1 + \eta_2 q(z)](q(z))^k + tz(q(z))^{k-1} q'(z), \end{aligned} \tag{2.8}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta} < q(z), \quad z \in U, f(z) \neq -z$$

and q is the best dominant.

Proof. Define the function p by

$$p(z) = \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta}, \quad z \in U, f(z) \neq -z. \tag{2.9}$$

Note that

$$\begin{aligned} [\eta_1 + \eta_2 p(z)](p(z))^k + tz(p(z))^{k-1} p'(z) = \eta_1 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} + \eta_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta(k+1)} \\ + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \end{aligned} \tag{2.10}$$

From (2.8) and (2.10), we have

$$[\eta_1 + \eta_2 p(z)](p(z))^k + tz(p(z))^{k-1} p'(z) < [\eta_1 + \eta_2 q(z)](q(z))^k + tz(q(z))^{k-1} q'(z). \tag{2.11}$$

By setting

$$\theta(w) = (\eta_1 + \eta_2 w)w^k \text{ and } \phi(w) = t w^{k-1}, w \neq 0,$$

we see that $\theta(w)$ is analytic in \mathbb{C} , $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = tz(q(z))^{k-1} q'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = [\eta_1 + \eta_2 q(z)](q(z))^k + tz(q(z))^{k-1} q'(z).$$

It is clear that $Q(z)$ is starlike univalent in U ,

$$Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = Re \left\{ 1 + \frac{\eta_1 k}{t} + \frac{\eta_2(k+1)}{t} q(z) + \frac{zq''(z)}{q'(z)} + (k-1) \frac{zq'(z)}{q(z)} \right\}. \tag{2.12}$$

From (2.7) and (2.12), we have

$$Re \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0.$$

Therefore by Lemma 1.1 , we get $p(z) < q(z)$. By using (2.9), we obtain the result .

Remark 2.1 . Taking $\eta_1 = k = 0, \eta_2 = \delta = t = 1$ in Theorem 2.2 , we obtain the result in corollary 2.1 .

By taking $q(z) = (1 + Az)/(1 + Bz)$ ($-1 \leq B < A \leq 1$) in the Theorem 2.2 , we obtain the following corollary :

Corollary 2.5. Let the function q be convex univalent in the unit disk U and assume that

$$Re \left\{ \frac{\eta_1 k}{t} + \frac{\eta_2 (k + 1)(1 + Az)}{t(1 + Bz)} + \frac{1 + k(A - B)z - ABz^2}{(1 + Az)(1 + Bz)} \right\} > 0.$$

If $f \in L(\lambda)$ satisfies

$$\begin{aligned} \eta_1 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} + \eta_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta(k+1)} + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \\ < \left[\eta_1 + \eta_2 \left(\frac{1 + Az}{1 + Bz} \right) \right] \left(\frac{1 + Az}{1 + Bz} \right)^k + \frac{t(A - B)(1 + Az)^{k-1}z}{(1 + Bz)^{k+1}}, \end{aligned}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant .

By taking $q(z) = e^{\gamma Az}$, $\gamma \neq 0$ in the Theorem 2.2 , we obtain the following corollary :

Corollary 2.6. Let the function q be convex univalent in the unit disk U and assume that

$$Re \left\{ 1 + k\gamma Az + \frac{\eta_1 k}{t} + \frac{\eta_2 (k + 1)}{t} e^{\gamma Az} \right\} > 0.$$

If $f \in L(\lambda)$ satisfies

$$\begin{aligned} \eta_1 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} + \eta_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta(k+1)} + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \\ < [\eta_1 + \eta_2 e^{\gamma Az}] e^{k\gamma Az} + t\gamma A z e^{k\gamma Az}, \end{aligned}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta} < e^{\gamma Az}, \quad \gamma \neq 0$$

and $q(z) = e^{\gamma Az}$ is the best dominant .

By taking $q(z) = \left[\frac{1+z}{1-z} \right]^\gamma$, $\gamma \neq 0$ in the Theorem 2.2 , we obtain the following corollary:

Corollary 2.7. Let the function q be convex univalent in the unit disk U and assume that

$$Re \left\{ \frac{\eta_1 k}{t} + \frac{\eta_2 (k+1)(1+z)^\gamma}{t(1-z)^\gamma} + \frac{z^2 + 2k\gamma z + 1}{(1-z)(1+z)} \right\} > 0.$$

If $f \in L(\lambda)$ satisfies

$$\begin{aligned} \eta_1 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} + \eta_2 \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta(k+1)} + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^{\delta k} \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] \\ < \left[\eta_1 + \eta_2 \left(\frac{1+z}{1-z} \right)^\gamma \right] \left(\frac{1+z}{1-z} \right)^{k\gamma} + \frac{2t\gamma(1+z)^{k\gamma-1}z}{(1-z)^{k\gamma+1}}, \end{aligned}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < \left[\frac{1+z}{1-z} \right]^\gamma, \quad \gamma \neq 0$$

and $q(z) = \left[\frac{1+z}{1-z} \right]^\gamma$ is the best dominant .

Theorem 2.3. Let the function q be convex univalent in the unit disk U , $q(z) \neq 0$ and assume that

$$Re \left\{ 1 + \frac{zq''(z)}{q'(z)} + \frac{1}{t} \right\} > 0, \tag{2.13}$$

where $t \in \mathbb{C} \setminus \{0\}$. If $f \in T(\lambda)$ satisfies

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] < q(z) + tzq'(z), \tag{2.14}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < q(z), \quad z \in U, f(z) \neq -z$$

and q is the best dominant .

Proof . Define the function p by

$$p(z) = \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta, \quad z \in U, f(z) \neq -z. \tag{2.15}$$

Note that

$$p(z) + tzp'(z) = \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right]. \tag{2.16}$$

From (2.14) and (2.16) , we have

$$p(z) + tzp'(z) < q(z) + tzq'(z). \tag{2.17}$$

By setting $\psi = 1$ in Lemma 1.2 , we get $p(z) < q(z)$.By using (2.15) , we obtain the result .

By taking $q(z) = (1 + Az)/(1 + Bz)$ ($-1 \leq B < A \leq 1$) in the Theorem 2.3 , we obtain the following corollary :

Corollary 2.8. Let the function q be convex univalent in the unit disk U and assume that

$$Re \left\{ \frac{1 - Bz}{1 + Bz} + \frac{1}{t} \right\} > 0.$$

If $f \in T(\lambda)$ satisfies

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] < \frac{1 + Az}{1 + Bz} + \frac{t(A - B)z}{(1 + Bz)^2},$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant .

By taking $q(z) = e^{\gamma Az}$, $\gamma \neq 0$ in the Theorem 2.3 , we obtain the following corollary :

Corollary 2.9. Let the function q be convex univalent in the unit disk U and assume that

$$\operatorname{Re} \left\{ 1 + \gamma Az + \frac{1}{t} \right\} > 0.$$

If $f \in T(\lambda)$ satisfies

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] < e^{\gamma Az} + t\gamma Aze^{\gamma Az},$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < e^{\gamma Az}, \quad \gamma \neq 0$$

and $q(z) = e^{\gamma Az}$ is the best dominant .

By taking $q(z) = \left[\frac{1+z}{1-z} \right]^\gamma$, $\gamma \neq 0$ in the Theorem 2.3 , we obtain the following corollary :

Corollary 2.10. Let the function q be convex univalent in the unit disk U and assume that

$$\operatorname{Re} \left\{ \frac{z^2 + 2\gamma z + 1}{(1 - z)(1 + z)} + \frac{1}{t} \right\} > 0.$$

If $f \in T(\lambda)$ satisfies

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta + t\delta \left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)} \right] < \left[\frac{1 + z}{1 - z} \right]^\gamma + \frac{2t\gamma(1 + z)^{\gamma-1}z}{(1 - z)^{\gamma+1}},$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)} \right]^\delta < \left[\frac{1 + z}{1 - z} \right]^\gamma, \quad \gamma \neq 0$$

and $q(z) = \left[\frac{1+z}{1-z} \right]^\gamma$ is the best dominant .

3. Applications

We introduce some applications containing fractional derivative operators . Assume that

$$g(z) = \sum_{n=2}^{\infty} \sigma_n z^n.$$

By Definition 1.2 , we have

$$D_z^\lambda g(z) = \sum_{n=2}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} \sigma_n z^{n-\lambda} = \sum_{n=2}^{\infty} a_n z^{n-\lambda},$$

where

$$a_n = \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \lambda)} \sigma_n, \quad n = 2, 3, \dots$$

Thus $z + D_z^\lambda g(z) \in L(\lambda)$ and $z - D_z^\lambda g(z) \in T(\lambda)$ ($\sigma_n \geq 0$), then we have the following results :

Theorem 3.1. Let the assumptions of Theorem 2.1 hold .Then

$$\left[\frac{2z + zD_z^\lambda g'(z)}{2z + D_z^\lambda g(z)} \right]^\delta < q(z),$$

and q is the best dominant .

Proof . Define the function f by

$$f(z) = z + D_z^\lambda g(z) \quad (z \in U, f(z) \neq -z),$$

it can easily observed that $f \in L(\lambda)$. Since $g(0) = 0$, then $[D_z^\lambda g(z)]' = D_z^\lambda g'(z)$.

Thus by using Theorem 2.1 , we obtain the result .

Theorem 3.2. Let the assumptions of Theorem 2.2 hold .Then

$$\left[\frac{2z + zD_z^\lambda g'(z)}{2z + D_z^\lambda g(z)} \right]^\delta < q(z),$$

and q is the best dominant .

Proof . The proof is similar to that of Theorem 3.1 .

Theorem 3.3. Let the assumptions of Theorem 2.3 hold .Then

$$\left[\frac{2z - zD_z^\lambda g'(z)}{2z - D_z^\lambda g(z)} \right]^\delta < q(z),$$

and q is the best dominant .

Proof . Define the function f by

$$f(z) = z - D_z^\lambda g(z) \quad (z \in U, f(z) \neq -z),$$

it can easily observed that $f \in T(\lambda)$. Since $g(0) = 0$, then $[D_z^\lambda g(z)]' = D_z^\lambda g'(z)$.

Thus by using Theorem 2.3 , we obtain the result .

By using (1.10) , we have

$$J_{0,z}^{\lambda,\mu,\nu} g(z) = \sum_{n=2}^{\infty} \frac{\Gamma(n + 1)\Gamma(n - \mu + \nu + 1)}{\Gamma(n - \mu + 1)\Gamma(n - \lambda + \nu + 1)} \sigma_n z^{n-\mu} = \sum_{n=2}^{\infty} a_n z^{n-\mu},$$

where

$$a_n = \frac{\Gamma(n + 1)\Gamma(n - \mu + \nu + 1)}{\Gamma(n - \mu + 1)\Gamma(n - \lambda + \nu + 1)} \sigma_n, \quad n = 2, 3, \dots$$

Let $\mu = \lambda$. Then $z + J_{0,z}^{\lambda,\mu,\nu} g(z) \in L(\lambda)$ and $z - J_{0,z}^{\lambda,\mu,\nu} g(z) \in T(\lambda)$ ($\sigma_n \geq 0$), thus we have the following results :

Theorem 3.4. Let the assumptions of Theorem 2.1 hold .Then

$$\left[\frac{z \left(2z + J_{0,z}^{\lambda,\mu,\nu} g(z) \right)'}{2z + J_{0,z}^{\lambda,\mu,\nu} g(z)} \right]^\delta < q(z),$$

and q is the best dominant .

Proof . Define the function f by

$$f(z) = z + J_{0,z}^{\lambda,\mu,\nu} g(z) \quad (z \in U, f(z) \neq -z),$$

it can easily observed that $f \in L(\lambda)$. Thus by using Theorem 2.1 , we obtain the result .

Theorem 3.5. Let the assumptions of Theorem 2.2 hold .Then

$$\left[\frac{z \left(2z + J_{0,z}^{\lambda,\mu,\nu} g(z) \right)'}{2z + J_{0,z}^{\lambda,\mu,\nu} g(z)} \right]^{\delta} < q(z) ,$$

and q is the best dominant .

Proof . The proof is similar to that of Theorem 3.4 .

Theorem 3.6. Let the assumptions of Theorem 2.3 hold .Then

$$\left[\frac{z \left(2z - J_{0,z}^{\lambda,\mu,\nu} g(z) \right)'}{2z - J_{0,z}^{\lambda,\mu,\nu} g(z)} \right]^{\delta} < q(z) ,$$

and q is the best dominant .

Proof . Define the function f by

$$f(z) = z - J_{0,z}^{\lambda,\mu,\nu} g(z) \quad (z \in U, f(z) \neq -z) ,$$

it can easily observed that $f \in T(\lambda)$. Thus by using Theorem 2.3 , we obtain the result .

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