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Some Properties of a New Certain Class of Multivalent Functions with Negative Coefficients

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Abstract

In the present paper, we study a new class $A_{p,0}(A,B,\alpha)$ of multivalent functions with negative coefficients in the unit disk $U=\{z:|z|<1\}$, we obtain some properties like coefficient bounds, distortion bounds, radii of close-to-convexity, starlikeness and convexity of order $\delta\ (0\le\delta<p)$ for our class. We also obtain several results, like, arithmetic mean, convex linear combinations, extreme points and integral operators for the class $A_{p,0}(A,B,\alpha)$.

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1. Introduction

Let $A_p(k)$ denote the class of function, $f(z)$ of the form:

$$f(z)=z^p+\sum_{k=n+p}^{\infty}a_k z^k, \quad a_k>0, \quad n,p\in IN=\{1,2,3,...\} \tag{1.1}$$

Which are analytic and multivalent in the open unit disk
$U=\{z: z\in C \text{ and } |z|<1\}$.

A function f belonging to the class $A_p(k)$ is said to be in the class $A_{p,m}(A,B,\alpha)$ if and only if

$$p+\operatorname{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\}>0 \text{ for } z\in U .$$

In the other words, $f\in A^*_{p,m}(A,B,\alpha)$ if and only if it satisfies the condition

$$\left|\frac{\dfrac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\dfrac{f^{(p-1)}(z)}{zf^{(p)}(z)}+1}{(A-B)(p-\alpha)+pB-B\left[p+\dfrac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]}\right|<1$$

where $-1\le B<A\le 1$, $-1\le B<0$ and $0\le\alpha<p$. Let $A_{p,m}$ denote the subclass of $A_p(k)$ consisting of functions analytic and multivalent which can be expressed in the form:

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_k z^k,\quad a_k\ge 0. \tag{1.2}$$

Let us define

$$A_{p,0}(A,B,\alpha)=A_{p,m}(A,B,\alpha)\bigcap A_{p,m}$$

In the present paper, we obtain coefficient bounds, distortion bounds, radii of close-to-convexity, starlikeness and convexity, integral operators, closure theorems. Another classes studied by some authors like Nunokawa [2], Guneny and summer Eker [1], Zhi-Gang Wang and Neng Xu [4] and G. Oros [3].

2. Coefficient Bounds

Theorem 2.1: Let the function f is defined by (1.1). Then $f\in A_{p,0}(A,B,\alpha)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]a_k}{(k-p+1)!}\le (A-B)(p-\alpha)p! \tag{2.1}$$

The result is sharp.

Proof: Suppose that the inequality (2.1) holds true and let $|z|=1$. Then we obtain

$$\left|z^2 f^{(p+1)}(z)+zf^{(p)}(z)-f^{(p-1)}(z)\right|-\left|(A-B)(p-\alpha)zf^{(p)}(z)-Bz^2 f^{(p+1)}(z)\right|$$

$$=\left|-\sum_{k=n+p}^{\infty}\frac{(k-p)(k-p+2)k!}{(k-p+1)!}a_k z^{k-p+1}\right|-\left|(A-B)(p-\alpha)p!z-(A-B)(p-\alpha)\sum_{k=n+p}^{\infty}\frac{(k-p+1)k!}{(k-p+1)!}a_k z^{k-p+1}+B\sum_{k=n+p}^{\infty}\frac{(k-p)(k-p+1)k!}{(k-p+1)!}a_k z^{k-p+1}\right|$$

$$\le\sum_{k=n+p}^{\infty}\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!}a_k-(A-B)(p-\alpha)p!\le 0$$

by hypothesis. Hence, by the maximum modulus theorem, we have $f\in A_{p,0}(A,B,\alpha)$. Conversely, suppose that

$$\left|\frac{\dfrac{zf^{(p+1)}(z)}{f^{(p)}(z)}-\dfrac{f^{(p-1)}(z)}{zf^{(p)}(z)}+1}{(A-B)(p-\alpha)+pB-B\left[p+\dfrac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]}\right|$$

$$=\left|\frac{-\displaystyle\sum_{k=n+p}^{\infty}\frac{(k-p)(k-p+2)k!}{(k-p+1)!}a_k z^{k-p+1}}{(A-B)(p-\alpha)\left(p!z-\displaystyle\sum_{k=n+p}^{\infty}\frac{(k-p+1)k!}{(k-p+1)!}a_k z^{k-p+1}\right)+B\displaystyle\sum_{k=n+p}^{\infty}\frac{(k-p)(k-p+1)k!}{(k-p+1)!}a_k z^{k-p+1}}\right|<1$$

Since $\mathrm{Re}(z)\le|z|$ for all z, we have

$$\operatorname{Re}\left\{\frac{\sum_{k=n+p}^{\infty}\frac{(k-p)(k-p+2)k!}{(k-p+1)!}a_k z^{k-p+1}}{(A-B)(p-\alpha)\left(p!z-\sum_{k=n+p}^{\infty}\frac{(k-p+1)k!}{(k-p+1)!}a_k z^{k-p+1}\right)+B\sum_{k=n+p}^{\infty}\frac{(k-p)(k-p+1)k!}{(k-p+1)!}a_k z^{k-p+1}}\right\}<1$$

Choosing values of z on the real axis and letting $z\to 1^-$ through real values, we obtain

$$\sum_{k=n+p}^{\infty}\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!}a_k\le (A-B)(p-\alpha)p!,$$

which obviously is required assertion (2.1). Finally, the result (2.1) is sharp for the function

$$f(z)=z^p-\frac{(k-p+1)!(A-B)(p-\alpha)p!}{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}z^k \qquad (2.2)$$

Corollary 2.2: Let $f\in A_{p,0}(A,B,\alpha)$. Then

$$a_k\le\frac{(k-p+1)!(A-B)(p-\alpha)p!}{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]} \qquad (2.3)$$

The equality in (2.3) is attained for the function f given by (2.2).

3. Distortion Bounds

Theorem (3.1): Let $f\in A_{p,0}(A,B,\alpha)$. Then for $|z|=r<1$

$$r^p-\frac{(n+1)!(A-B)(p-\alpha)p!}{(n+p)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]}r^{n+p}\le|f(z)|$$

$$\le r^p+\frac{(n+1)!(A-B)(p-\alpha)p!}{(n+p)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]}r^{n+p} \qquad (3.1)$$

and

$$pr^{p-1}-\frac{(n+1)!(A-B)(p-\alpha)p!}{(n+p-1)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]}r^{n+p-1}\le|f'(z)|$$

$$\le pr^{p-1}+\frac{(n+1)!(A-B)(p-\alpha)p!}{(n+p-1)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]}r^{n+p-1} \qquad (3.2)$$

The inequalities (3.1) and (3.2) are sharp.

Proof: Let

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_k z^k,\quad a_k\ge 0.$$

From Theorem 2.1, we have

$$\frac{(n+p)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]}{(n+1)!}\sum_{k=n+p}^{\infty}a_k$$

$$\le\sum_{k=n+p}^{\infty}\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!}a_k\le (A-B)(p-\alpha)p!,$$

which

$$\sum_{k=n+p}^{\infty}a_k\le\frac{(n+1)!(A-B)(p-\alpha)p!}{(n+p)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]}$$

and

$$\sum_{k=n+p}^{\infty}ka_k\le\frac{(n+1)!(A-B)(p-\alpha)p!}{(n+p-1)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]}.$$

Consequently, for $|z| = r < 1$, we obtain

$$|f(z)| \le r^p + r^{n+p} \sum_{k=n+p}^{\infty} a_k$$

$$\le r^p + \frac{(n+1)!(A-B)(p-\alpha)p!}{(n+p)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]} r^{n+p}$$

and

$$|f(z)| \ge r^p - r^{n+p} \sum_{k=n+p}^{\infty} a_k$$

$$\ge r^p - \frac{(n+1)!(A-B)(p-\alpha)p!}{(n+p)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]} r^{n+p},$$

which prove that the assertion (3.1) of Theorem 3.1 holds. The inequality in (3.2) can be proved in a similar manner and we omit the details.

The bounds in (3.1) and (3.2) are attained for the function f given by

$$f(z) = z^p - \frac{(n+1)!(A-B)(p-\alpha)}{(n+p)!\left[n((n+2)-B(n+1))+(n+1)(A-B)(p-\alpha)\right]} z^{n+p}.$$

4. Radii of Close-to- Convexity, Starlikeness and Convexity

Theorem 4.1: Let $f \in A_{p,0}(A,B,\alpha)$. Then f is $p-$valent close- to- convex of order δ $(0 \le \delta < p)$ in $|z| < R_1$, where

$$R_1 = \inf_k \left\{ \left[\frac{(p-\delta)k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+)(A-B)(p-\alpha)\right]}{k(k-p+1)!(A-B)(p-\alpha)p!} \right]^{\frac{1}{k-p}} \right\}, \tag{4.1}$$

$k \ge n+p$.

Proof: We must show that $\left| \frac{f'(z)}{z^{p-1}} - p \right| \le p-\delta$ for $|z| < R_1$, we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le \sum_{k=n+p}^{\infty} k a_k |z|^{k-p}$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le p-\delta \text{ if } \sum_{k=n+p}^{\infty} \left(\frac{k}{p-\delta} \right) a_k |z|^{k-p} \le 1. \tag{4.2}$$

By Theorem 2.1, we have

$$\sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]a_k}{(k-p+1)!(A-B)(p-\alpha)p!} \le 1 \tag{4.3}$$

Hence (4.2) will be true if

$$\frac{k|z|^{k-p}}{p-\delta} \le \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!(A-B)(p-\alpha)p!}$$

equivalently if

$$|z| \le \left\{ \left[\frac{(p-\delta)k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{k(k-p+1)!(A-B)(p-\alpha)p!} \right]^{\frac{1}{k-p}} \right\}, \tag{4.4}$$

$k \ge n+p$. The theorem follows from (4.4).

Theorem 4.2: Let $f \in A_{p,0}(A,B,\alpha)$. Then f is $p-$valent starlike of order δ $(0 \le \delta < p)$ in $|z| < R_2$, we have

$$R_2 = \inf_k \left\{ \left[\frac{(p-k)k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+)(A-B)(p-\alpha)\right]}{(k-\delta)(k-p+1)!(A-B)(p-\alpha)p!} \right]^{\frac{1}{k-p}} \right\}, \tag{4.5}$$

Proof: It is sufficient to show that $\left|\frac{zf^{'}(z)}{f(z)} - p\right| \le p-\delta$ for $|z| < R_2$, we have

$$\left|\frac{zf^{'}(z)}{f(z)} - p\right| \le \frac{\sum_{k=n+p}^{\infty}(k-p)a_k|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}a_k|z|^{k-p}}.$$

Thus

$$\left|\frac{zf^{'}(z)}{f(z)} - p\right| \le p-\delta \text{ if } \sum_{k=n+p}^{\infty}\left(\frac{k-\delta}{p-\delta}\right)a_k|z|^{k-p} \le 1. \tag{4.6}$$

By using (4.3) and (4.6) will be true if

$$\frac{k-\delta}{p-\delta}|z|^{n-p} \le \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!(A-B)(p-\alpha)p!}$$

equivalent if

$$|z| \le \left[\frac{(p-\delta)k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p)(k-p+1)!(A-B)(p-\alpha)p!} \right]^{\frac{1}{k-p}}, \tag{4.7}$$

$k \ge n+p$. The theorem follows from (4.7).

Theorem 4.3: Let $f \in A_{p,0}(A,B,\alpha)$. Then f is $p-$valent convex function of order δ $(0 \le \delta < p)$ in $|z| < R_3$, we have

$$R_3 = \inf_k \left\{ \left[\frac{p(p-\delta)k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{k(k-p)(k-p+1)!(A-B)(p-\alpha)p!} \right]^{\frac{1}{k-p}} \right\}, \tag{4.8}$$

Proof: We must show that $\left|\left[1+\frac{zf^{''}(z)}{f^{'}(z)}\right] - p\right| \le p-\delta$ for $|z| < R_3$, we have

$$\left|\left[1+\frac{zf^{''}(z)}{f^{'}(z)}\right] - p\right| \le \frac{\sum_{k=n+p}^{\infty}k(k-p)a_k|z|^{k-p}}{p-\sum_{k=n+p}^{\infty}ka_k|z|^{k-p}}$$

Thus

$$\left|\left[1+\frac{zf^{''}(z)}{f^{'}(z)}\right] - p\right| \le p-\delta \text{ if } \sum_{k=n+p}^{\infty}\frac{k(k-p)}{p(p-\delta)}a_k|z|^{k-p} \le 1 \tag{4.9}$$

By using (4.3) and (4.4) will be true if

$$\frac{k(k-p)}{p(p-\delta)}|z|^{k-p} \le \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!(A-B)(p-\alpha)p!}$$

equivalently if

$$|z| \leq \left[\frac{p(p-\delta)k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{k(k-p)(k-p+1)!(A-B)(p-\alpha)p!} \right]^{\frac{1}{k-p}}, \tag{4.10}$$

$k \geq p+n$. The theorem follows from (4.10).

Remark (4.4): The results in the Theorem 4.1 , 4.2 and 4.3 are sharp with the extremal function f given by (2.2). Furthermore, take $\delta = 0$ in the Theorems 4.1 ,4.2 and 4.3, we obtain radian of close-to-convexity, starlikeness and convexity respectively.

5. Integral Operators

Let c be a real number such that $c > -p$. If $f \in A_{p,0}(A,B,\alpha)$, then the function F defined by

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt \tag{5.1}$$

also belong to $A_{p,0}(A,B,\alpha)$.

Proof: Let

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

Then from the representation of F, it follows that

$$F(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k ,$$

where $b_k = \left(\frac{c+p}{c+k}\right) a_k$. Therefore using Theorem 2.1for the coefficients of F, we have

$$\sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!} b_k$$

$$= \sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!} \left(\frac{c+p}{c+k}\right) a_k \leq (A-B)(p-\alpha)p!,$$

Since $\frac{c+p}{c+k} < 1$ and $f \in A_{p,0}(A,B,\alpha)$. Hence $F \in A_{p,0}(A,B,\alpha)$.

Theorem 5.2: Let c be a real number such that $c > -p$. If $F \in A_{p,0}(A,B,\alpha)$, then the function f defined by (5.1) is $p-$valent in $|z| < R^*$, where

$$R^* = \inf_k \left\{ \left[\frac{p(c+p)k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{k(c+k)(k-p+1)!(A-B)(p-\alpha)p!} \right]^{\frac{1}{k-p}} \right\} \tag{5.2}$$

The result is sharp. Sharpness follows if we take

$$f(z) = z^p - \left(\frac{c+k}{c+p}\right) \frac{(k-p+1)!(A-B)(p-\alpha)p!}{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]} z^k .$$

Proof: Let $F(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k = \frac{c+p}{z} \int_0^z t^{c-1} f(t)dt$ so

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \left(\frac{c+k}{c+p}\right) b_k z^k , \quad c > -p .$$

Thus it is sufficient to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p$, $|z| < r$. But

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|\sum_{k=n+p}^{\infty}k\left(\frac{c+k}{c+p}\right)b_k z^{k-p}\right|, \text{ then}$$

$$\sum_{k=n+p}^{\infty}\frac{k}{p}\left(\frac{c+k}{c+p}\right)b_k|z|^{k-p}\le 1 \tag{5.3}$$

Since $F\in A_{p,0}(A,B,\alpha)$, then by (2.1) we have

$$\sum_{k=n+p}^{\infty}\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!(A-B)(p-\alpha)p!}b_k\le 1.$$

Therefore (5.3) will be true if

$$\frac{k}{p}\left(\frac{c+k}{c+p}\right)|z|^{k-p}\le\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!(A-B)(p-\alpha)p!}$$

or

$$|z|\le\left[\frac{p(c+p)k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{k(c+k)(k-p+1)!(A-B)(p-\alpha)p!}\right]^{\frac{1}{k-p}},$$

and this proves result. Sharpness of this theorem follows if we put

$$f(z)=z^p-\left(\frac{c+k}{c+p}\right)\frac{(k-p+1)!(A-B)(p-\alpha)p!}{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}z^k .$$

6. Closure Theorems

In the next theorems, we show that the class $A_{p,0}(A,B,\alpha)$ is closed under " arithmetic mean" and " convex linear combinations".

Theorem 6.1: Let $f_j(z)=z^p-\sum_{k=n+p}^{\infty}a_{k,j}z^k$, $j=1,2,\ldots$ and $h(z)=z^p-\sum_{k=n+p}^{\infty}b_k z^k$, where $b_k=\sum_{j=1}^{\infty}\lambda_j a_{k,j}$, $\lambda_j>0$ and $\sum_{j=1}^{\infty}\lambda_j=1$. If $f_j\in A_{p,0}(A,B,\alpha)$ for each $j=1,2,3,\ldots$, then $h\in A_{p,0}(A,B,\alpha)$.

Proof: If $f_j\in A_{p,0}(A,B,\alpha)$, then we have from Theorem 2.1 that

$$\sum_{k=n+p}^{\infty}\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!}a_{k,j}\le(A-B)(p-\alpha)p!,\quad j=1,2,3,\ldots$$

Therefore

$$\sum_{k=n+p}^{\infty}\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!}b_k$$

$$=\sum_{k=n+p}^{\infty}\frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!}\left(\sum_{j=1}^{\infty}\lambda_j a_{k,j}\right)\le(A-B)(p-\alpha)p!.$$

Hence by Theorem 2.1, $h\in A_{p,0}(A,B,\alpha)$.

Theorem 6.2: The class $A_{p,0}(A,B,\alpha)$ is closed under convex linear combinations.

Proof: Let the functions $f(z)=z^p-\sum_{k=n+p}^{\infty}a_k z^k$ and $g(z)=z^p-\sum_{k=n+p}^{\infty}b_k z^k$ belong to the class $A_{p,0}(A,B,\alpha)$. We must show that the function h defined by

$$h(z)=(1-\lambda)f(z)+\lambda g(z)$$

belong to $A_{p,0}(A,B,\alpha)$ for $0 \le \lambda \le 1$.

Since f and g are in the class $A_{p,0}(A,B,\alpha)$, then

$$\sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!} a_k \le (A-B)(p-\alpha)p!$$

and

$$\sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!} b_k \le (A-B)(p-\alpha)p!$$

Therefore for $0 \le \lambda \le 1$, we have

$$h(z) = z^p - \sum_{k=n+p}^{\infty} [(1-\lambda)a_k + \lambda b_k] z^k .$$

By Theorem 2.1, we have

$$\sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!} [(1-\lambda)a_k + \lambda b_k]$$

$$= (1-\lambda) \sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!} a_k$$

$$+ \lambda \sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!} b_k$$

$$\le (1-\lambda)(A-B)(p-\alpha)p! + \lambda(A-B)(p-\alpha)p! = (A-B)(p-\alpha)p!,$$

which implies that $h \in A_{p,0}(A,B,\alpha)$ and this completes the proof.

Theorem 6.3: Let $f_{n+p-1}(z) = z^p$ and

$$f_k(z) = z^p - \frac{(k-p+1)!(A-B)(p-\alpha)p!}{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]} z^k , \quad k \ge n+p .$$

Then $f \in A_{p,0}(A,B,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=n+p-1}^{\infty} \lambda_k f_k(z), \quad z \in U,$$

where $\lambda_k \ge 0,\ \lambda_{n+p-1} = 1 - \sum_{k=n+p}^{\infty} \lambda_k$.

Proof: Let us suppose that

$$f(z) = \sum_{k=n+p-1}^{\infty} \lambda_k f_k(z)$$

$$= \lambda_{n+p-1} f_{n+p-1}(z) + \sum_{k=n+p}^{\infty} \lambda_k \left(z^p - \frac{(k-p+1)!(A-B)(p-\alpha)p!}{k![(k-p)((k-p+2)-B(k-p+1))+(k-p+1))+(k-p+1)(A-B)(p-\alpha)]} z^k \right)$$

$$= \left(\lambda_{n+p-1} + \sum_{k=n+p}^{\infty} \lambda_k \right) z^p - \sum_{k=n+p}^{\infty} \frac{(k-p+1)!(A-B)(p-\alpha)p!}{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]} \lambda_k z^k$$

Then from Theorem 2.1, we have

$$= \sum_{k=n+p}^{\infty} \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!}$$

$$\times \frac{(k-p+1)!(A-B)(p-\alpha)p!}{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]} \lambda_k \le (A-B)(p-\alpha)p!.$$

Hence $f \in A_{p,0}(A,B,\alpha)$. Conversely, let $f \in A_{p,0}(A,B,\alpha)$. It follows from Corollary 2.2 that

$$a_k \le \frac{(k-p+1)!(A-B)(p-\alpha)p!}{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}$$

Setting

$$\lambda_k = \frac{k!\left[(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)\right]}{(k-p+1)!(A-B)(p-\alpha)p!}a_k, \quad k \ge n+p$$

and $\lambda_{n+p-1} = 1 - \sum_{k=n+p}^{\infty}\lambda_k$, we have

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$= z^p - \sum_{k=n+p}^{\infty}\lambda_k z^p + \sum_{k=n+p}^{\infty}\lambda_k z^p$$

$$-\sum_{k=n+p}^{\infty}\lambda_k \frac{(k-p+1)!(A-B)(p-\alpha)p!}{k![(k-p)((k-p+2)-B(k-p+1))+(k-p+1)(A-B)(p-\alpha)]}z^k$$

$$= \lambda_{n+p-1}f_{n+p-1}(z) + \sum_{k=n+p}^{\infty}\lambda_k f_k(z) = \sum_{k=n+p-1}^{\infty}\lambda_k f_k(z).$$

This completes the proof of the Theorem 6.3.

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