Certain results on a class of multivalent functions of power order defined by a linear integral operator

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Abstract .

By using a linear integral operator , a class of multivalent functions of power order is introduced. Some important results of this class such as coefficient estimates , integral means inequalities and other property are found.

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1. Introduction

Let $\sum_{p,\eta}$ be the class of functions f of the form:

$$f(z)=z^{p+\eta}+\sum_{k=2}^{\infty}a_k z^{k+p+\eta},\ (0\le\eta<1,a_k\ge 0) \quad (1)$$

that are analytic in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.

Let

$$g(z)=z^{p+\eta}+\sum_{k=2}^{\infty}b_k z^{k+p+\eta},(0\le\eta<1,b_k\ge 0,z\in U). \quad (2)$$

A convolution (orHadamard product) of two power series f of the form (1) and a function g of the form (2) is defined by :

$$(f*g)(z)=z^{p+\eta}+\sum_{k=2}^{\infty}a_k b_k z^{k+p+\eta}=(g*f)(z),(z\in U).$$

Note that the authors defined and studied some classes of analytic functions take the form (1) in [1].

In this paper , we need to introduce a generalized integral operator such that class can be defined by means of this integral operator. For a function $f\in\sum_{p,\eta}$ given by (1), we define the integral operator $K_{c,p,\eta}^{\delta}(c>-(p+\eta))$ and $(0\le\eta<1,p\in\mathbb{N})$ by:

$$K_{c,p,\eta}^{\delta}f(z)=\frac{(c+p+\eta)^{\delta}}{\Gamma(\delta)z^c}\int_0^z t^{c-1}\left(log\frac{z}{t}\right)^{\delta-1}f(t)dt$$
$$=z^{p+\eta}+\sum_{k=2}^{\infty}\left[\frac{c+p+\eta}{c+k+p+\eta}\right]^{\delta}a_k z^{k+p+\eta}, (3)$$

when $\eta=0$ the operator $K_{c,p,0}^{\delta}$ was introduced by Komatu [7].

Clearly , (3) yields :

$$f \in \sum_{p,\eta} \Rightarrow \mathrm{K}^{\delta}_{c,p,\eta} f \in \sum_{p,\eta}.$$

A function $f \in \sum_{p,\eta}$ is said to be $(p+\eta)-$ valent starlike of order $\rho, (0 \le \rho < p+\eta)$ if and only if :

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho\,, \quad (z \in U).$$

Let $S_{p,\eta}(\rho)$ denote the class of all those functions.

A function $f \in \sum_{p,\eta}$ is said to be $(p+\eta)-$ valent convex of order $\rho, (0 \le \rho < p+\eta)$ if and only if :

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho\,, \quad (z \in U).$$

Let $C_{p,\eta}(\rho)$ denote the class of all those functions.

A function $f \in \sum_{p,\eta}$ is said to be $(p+\eta)-$ valent close $-$ to $-$ convex of order $\rho, (0 \le \rho < p+\eta)$ if and only if :

$$Re\left\{\frac{f'(z)}{z^{(p+\eta)-1}}\right\} > \rho\,, \quad (z \in U).$$

Let $K_{p,\eta}(\rho)$ denote the class of all those functions.

We suppose $L(p,\eta,\delta)$ denote the subclass of $\sum_{p,\eta}$ consisting of functions f which satisfy :

$$\left|\frac{1}{\beta}\left[\frac{\mu z\left(\mathrm{K}^{\delta}_{c,p,\eta} f'(z)\right)' + (1-\mu)\left(\mathrm{K}^{\delta}_{c,p,\eta} f(z)\right)'}{\mu\left(\mathrm{K}^{\delta}_{c,p,\eta} f'(z)\right) + (1-\mu) z^{(p+\eta)-1}} - (p+\eta)\right]\right| < \gamma\,, \qquad (4)$$

where $\delta > 0, c > -(p+\eta), 0 \le \eta < 1, \beta \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*$,$0 < \gamma \le p+\eta$ and $\mathrm{K}^{\delta}_{c,p,\eta} f$ given by (3).

2. Coefficient estimates

Theorem 1.

$$\left|\frac{1}{\beta}\left[\frac{\mu z\left(\mathrm{K}^{\delta}_{c,p,\eta} f'(z)\right)' + (1-\mu)\left(\mathrm{K}^{\delta}_{c,p,\eta} f(z)\right)'}{\mu\left(\mathrm{K}^{\delta}_{c,p,\eta} f'(z)\right) + (1-\mu) z^{(p+\eta)-1}} - (p+\eta)\right]\right| < \gamma\,, (z \in U).$$

Let the function $f \in \sum_{p,\eta}$ be defined by (1). Then $f \in L(p,\eta,\delta)$ if and only if

$$\sum_{k=2}^{\infty} a_k \frac{(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]}{\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)}\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} \le 1, \qquad (5)$$

where $\delta > 0, c > -(p+\eta), 0 \le \eta < 1, \beta \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*$ and $0 < \gamma \le p+\eta$.

Proof. Assume that inequality (5) holds true and $|z| = 1$. Then ,we obtain

$$|\mu z\left(\mathrm{K}^{\delta}_{c,p,\eta} f'(z)\right)' + (1-\mu)\left(\mathrm{K}^{\delta}_{c,p,\eta} f(z)\right)'$$

$$-(p+\eta)\,\mu\left(\mathrm{K}^{\delta}_{c,p,\eta} f'(z)\right) - (1-\mu)(p+\eta) z^{(p+\eta)-1}|$$

$$-\gamma\,|\beta|\left|\mu\left(\mathrm{K}^{\delta}_{c,p,\eta} f'(z)\right) + (1-\mu) z^{(p+\eta)-1}\right|$$

$$= |-\mu(p+\eta) z^{(p+\eta)-1}$$

$$+\sum_{k=2}^{\infty}(k+p+\eta)\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}[\mu(k-1) + (1-\mu)] a_k z^{k+p+\eta-1}|$$

$$-\gamma|\beta||\left(\mu(p+\eta)+(1-\mu)\right) z^{(p+\eta)-1}$$

$$+\sum_{k=2}^{\infty}\mu(k+p+\eta)\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_k z^{k+p+\eta-1}|$$

$$\le \sum_{k=2}^{\infty}(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]$$

$$\times\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_k$$

$$\le \gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)\,,$$

by hypothesis.

Hence , by maximum modulus principle , we have $f \in L(p,\eta,\delta)$. Conversely , let $f \in L(p,\eta,\delta)$. Then

That is

$$= |-\mu(p+\eta) z^{(p+\eta)-1}$$

$$+\sum_{k=2}^{\infty} (k+p+\eta)\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}[\mu(k-1)+(1-\mu)]a_k z^{k+p+\eta-1}|$$

$$\div |(\mu(p+\eta)+(1-\mu))z^{(p+\eta)-1}$$

$$+\sum_{k=2}^{\infty} \mu(k+p+\eta)\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_k z^{k+p+\eta-1}|$$

$$<\gamma|\beta| , \qquad (6)$$

since $|Ref(z)| \le |f(z)|$ for all z , we have

$$| Re\{[-\mu(p+\eta)z^{(p+\eta)-1}$$

$$+\sum_{k=2}^{\infty} (k+p+\eta)\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}[\mu(k-1)+(1-\mu)]$$

$$\times a_k z^{k+p+\eta-1}]$$

$$\div [(\mu(p+\eta)+(1-\mu))z^{(p+\eta)-1}$$

$$+\sum_{k=2}^{\infty} \mu(k+p+\eta)\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_k z^{k+p+\eta-1}]\}|$$

$$<\gamma|\beta| , \qquad (7)$$

choosing z on real axis and allowing $z \to 1$, we have

$$\frac{\mu(p+\eta)+\sum_{k=2}^{\infty} (k+p+\eta)\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}[\mu(k-1)+(1-\mu)]a_k}{(\mu(p+\eta)+(1-\mu))-\sum_{k=2}^{\infty} \mu(k+p+\eta)\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_k}$$

$$\le \gamma|\beta|$$

which gives (5).

Finally , the result is sharp with extremal function f given by :

$$f(z) = z^{p+\eta}$$

$$+\frac{\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)}{(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}} z^{k+p+\eta} , \quad (8)$$

$k \ge 2.$

Corollary 1. Let the function f defined by (1) be in the class $L(p,\eta,\delta)$. Then

$$a_k \le \frac{\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)}{(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}} ,$$

$k \ge 2. \qquad (9)$

The equality in (9) is attained for the function f given by (8).

Definition 1. Let $0 \le \lambda < 1$ and $\xi, \zeta \in \mathbb{R}$. Then in terms of Guass's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator of order λ of the function f is defined by:

$$D_{0,z}^{\lambda,\xi,\zeta} f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\times\left\{z^{\lambda-\xi}\int_0^z (z-t)^{-\lambda}\, {}_2F_1\left(\xi-\lambda, 1-\zeta; 1-\lambda; 1-\frac{t}{z}\right) f(t)dt\right\}, \qquad (10)$$

where the function f is analytic in a simply connected region of the z - plane containing the origin , with the order

$f(z) = O(|z|)^{\epsilon}$, $(z \to 0)$ for $\epsilon > \max\{0, \xi-\zeta\}-1$ and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Note that

$$D_{0,z}^{\lambda,\lambda,\zeta} f(z) = D_z^{\lambda} f(z) \ , (0 \le \lambda < 1), \qquad (11)$$

where

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^{\lambda}}\, dt \ , \quad (0 \le \lambda < 1) \qquad (12)$$

is the Riemann – Liouville fractional derivative of order λ.

Definition 2. Under the hypothesis of Definition 1 , the fractional derivative of order $n+\lambda$ is defined , for a function f , by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \quad (0 \le \lambda < 1; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

It readily follows from Definition 1 that

$$D_z^{\lambda} z^{k+p+\eta} = \frac{\Gamma(k+p+\eta+1)}{\Gamma(k+p+\eta-\lambda+1)} z^{k+p+\eta-\lambda}, (0 \le \lambda < 1), \qquad (13)$$

we shall need the concept of subordination between analytic functions and subordination theorem of Littlewood [2].

Definition 3. If the functions f and g are analytic in the open unit disk U, then f is said to be subordinate to g in U if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) (z \in U)$. We denote this subordination by $f \prec g$.

Theorem 2.

If the function f and g are analytic in U with $f \prec g$, then

$$\int_0^{2\pi} \left|f(re^{i\vartheta})\right|^{\tau} d\vartheta \le \int_0^{2\pi} \left|g(re^{i\vartheta})\right|^{\tau} d\vartheta, (\tau > 0;\ 0 < r < 1). \qquad (14)$$

3. Integral means inequalities

Theorem 3.

Let g be of the form (2) and $f \in L(p,\eta,\delta)$ be of the form (1) and let for some $i \in \mathbb{N}$,

$$\frac{\varphi_i}{b_i} = \min_{k \ge 2} \frac{\varphi_k}{b_k},$$

where

$$\varphi_k = \frac{(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]}{\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)} \left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}.$$

Also , let for such $i \in \mathbb{N}$, the functions f_i and g_i be defined respectively by

$$f_i(z) = z^{p+\eta} + \frac{\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}} z^{i+p+\eta},$$

$$g_i(z) = z^{p+\eta} + b_i z^{i+p+\eta}. \qquad (15)$$

If there exists an analytic function w defined by

$$\{w(z)\}^i = \frac{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}}{[\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)]b_i} \sum_{k=2}^{\infty} a_k b_k z^k,$$

then , for $\tau > 0$ and $z = re^{i\vartheta} (0 < r < 1)$,

$$\int_0^{2\pi} |(f*g)(z)|^{\tau} d\vartheta \le \int_0^{2\pi} |(f_i * g_i)(z)|^{\tau} d\vartheta,$$

$(\tau > 0)$.

Proof. Convolution of f and g is defined as:

$$(f*g)(z) = z^{p+\eta} + \sum_{k=2}^{\infty} a_k b_k z^{k+p+\eta} = z^{p+\eta}\left(1 + \sum_{k=2}^{\infty} a_k b_k z^k\right).$$

Similarly , from (15) , we obtain

$$(f_i * g_i)(z) = z^{p+\eta} + \frac{[\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)]\, b_i}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}} z^{i+p+\eta}$$

$$= z^{p+\eta}\left(1 + \frac{[\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)]\, b_i}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}} z^{i}\right).$$

To prove the theorem , we must show that for $\tau > 0$ and $z = re^{i\vartheta}(0 < r < 1)$,

$$\int_0^{2\pi} \left|1 + \sum_{k=2}^{\infty} a_k b_k z^k\right|^{\tau} d\vartheta$$

$$\leq \int_0^{2\pi} \left|1 + \frac{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)]\, b_i}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}} z^i\right|^{\tau} d\vartheta.$$

Thus , by applying Theorem 2, it would suffice to show that

$$1 + \sum_{k=2}^{\infty} a_k b_k z^k \prec 1 + \frac{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)]\, b_i}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}} z^i. \quad (16)$$

If the subordination (16) holds true , then there exist an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \sum_{k=2}^{\infty} a_k b_k z^k = 1 + \frac{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)]\, b_i}{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}} \{w(z)\}^i.$$

From the hypothesis of the theorem , there exists an analytic function w given by

$$\{w(z)\}^i = \frac{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}}{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)]b_i} \times \sum_{k=2}^{\infty} a_k b_k z^k ,$$

which readily yields $w(0) = 0$. Thus for such function w , using the hypothesis in the coefficient inequality for the class $L(p,\eta,\delta)$, we get

$$|w(z)|^i \leq \frac{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}}{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)]b_i} \sum_{k=2}^{\infty} a_k b_k |z|^k$$

$$\leq |z|^2 \frac{(i+p+\eta)[1+\mu(i+\gamma|\beta|-2)]\left(\frac{c+p+\eta}{c+i+p+\eta}\right)^{\delta}}{[\gamma|\beta|(1-\mu) + (\gamma|\beta|-1)\mu(p+\eta)]b_i} \sum_{k=2}^{\infty} a_k b_k$$

$$\leq |z| < 1 .$$

Therefore ,the subordination (16) holds true , thus the theorem is proved.

Theorem 4.

Let $f \in L(p,\eta,\delta), p(z)$ be given by

$$p(z) = z^{p+\eta} + \sum_{s=1}^{m} b_{sj-(s-1)(p+\eta)} z^{sj-(s-2)(p+\eta)} , \quad (17)$$

$(p+\eta) > \lambda,\ j \geq 2 , m \geq 2$, and suppose that

$$\sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} a_k \leq \sum_{s=1}^{m} \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)\Gamma(3-\lambda-n+p+\eta)}{\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)\Gamma(2+p+\eta-\lambda)\Gamma(p+\eta-\lambda+1-n)} \times b_{sj-(s-1)(p+\eta)} \quad (18)$$

for $\lambda = 0$ or 1

$(0 \leq n, v < 1)$ and $2 \leq \lambda \leq k$ $(0 < n, v < 1)$,

where $(k+p+\eta-\lambda)_{\lambda+1}$ denotes the pochhammer symbol defined by

$(k+p+\eta-\lambda)_{\lambda+1} = (k+p+\eta-\lambda)(k+p+\eta-\lambda+1) \ldots (k+p+\eta)$.

Then for $z = re^{i\vartheta}(0 < r < 1)$,

$$\int_0^{2\pi} \left|D_z^{\lambda+n} f(z)\right|^{\tau} d\vartheta$$

$$\leq \int_0^{2\pi} \left| \frac{\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta-\lambda+1-n)} z^{v-n} D_z^{\lambda+v} p(z) \right|^\tau d\vartheta,$$

$(\tau > 0)$. (19)

Proof. By means of the fractional derivative formula (13) and Definition 2 , we find from (1) that

$$D_z^{\lambda+n} f(z) = \frac{\Gamma(p+\eta+1)}{\Gamma(p+\eta-\lambda+1-n)} z^{p+\eta-\lambda-n}$$

$$\times \left[1 + \sum_{n=2}^{\infty} \frac{\Gamma(k+p+\eta+1)\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)\Gamma(k+p+\eta-\lambda+1-n)} a_k z^k \right]$$

$$= \frac{\Gamma(p+\eta+1)}{\Gamma(p+\eta-\lambda+1-n)} z^{p+\eta-\lambda-n}$$

$$\times \left[1 + \sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} \frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)} \varphi(k) a_k z^k \right]$$

where
$\varphi(k) =$

$$\frac{\Gamma(k+p+\eta-\lambda)}{\Gamma(k+p+\eta+1-\lambda-n)} \begin{cases} \lambda = 0 \ or \ 1 & (0 \leq n < 1) \\ 2 \leq \lambda \leq k & (0 < n < 1) \end{cases},$$

$k \geq 2, k \in \mathbb{N}$.

Since $\varphi(k)$ is a decreasing function of k , we have

$$0 \leq \varphi(k) \leq \varphi(2) =$$

$$\frac{\Gamma(2+p+\eta-\lambda)}{\Gamma(3+p+\eta-\lambda-n)} \begin{cases} \lambda = 0 \ or \ 1 & (0 \leq n < 1) \\ 2 \leq \lambda \leq k & (0 < n < 1) \end{cases},$$

$k \geq 2, k \in \mathbb{N}$.

Similarly , by using (17) ,(13) and Definition 2 , we obtain

$$D_z^{\lambda+v} p(z) = \frac{\Gamma(p+\eta+1)}{\Gamma(p+\eta-\lambda+1-v)} z^{p+\eta-\lambda-v}$$

$$[1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}$$

$$\times b_{sj-(s-1)(p+\eta)} z^{sj-(s-1)(p+\eta)}]$$

Thus , we have

$$\frac{\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta-\lambda+1-n)} z^{v-n} D_z^{\lambda+v} p(z).$$

$$= \frac{\Gamma(p+\eta+1)}{\Gamma(p+\eta-\lambda+1-n)} z^{p+\eta-\lambda-n}$$

$$\times [1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}$$

$$\times b_{sj-(s-1)(p+\eta)} z^{sj-(s-1)(p+\eta)}]$$

For $z = re^{i\vartheta} (0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} \frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)} \varphi(k) a_k z^k \right|^\tau d\vartheta$$

$$\leq \int_0^{2\pi} |1 +$$

$$\sum_{s=1}^{m} \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}$$

$$\times b_{sj-(s-1)(p+\eta)} z^{sj-(s-1)(p+\eta)} |^\tau d\vartheta, (\tau > 0).$$

By applying Theorem 1. It suffices to show that

$$1 + \sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} \frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)} \varphi(k) a_k z^k$$

$$\prec 1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}$$

$$\times b_{sj-(s-1)(p+\eta)} z^{sj-(s-1)(p+\eta)}. \quad (20)$$

By setting

$$1 + \sum_{k=2}^{\infty} (k+p+\eta-\lambda)_{\lambda+1} \frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)} \varphi(k) a_k z^k$$

$$= 1 + \sum_{s=1}^{m} \frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}$$

$$\times b_{sj-(s-1)(p+\eta)} \{w(z)\}^{sj-(s-1)(p+\eta)}$$

we find that

$$\{w(z)\}^{sj-(s-1)(p+\eta)}$$

$$= \sum_{k=2}^{\infty}(k+p+\eta-\lambda)_{\lambda+1}\frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)}\varphi(k)a_k z^k$$

$$\times\frac{1}{\sum_{s=1}^{m}\frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}b_{sj-(s-1)(p+\eta)}}.$$

which readily yields $w(0)=0$. Therefore ,

we have

$$|w(z)|^{sj-(s-1)(p+\eta)} \leq$$

$$\sum_{k=2}^{\infty}(k+p+\eta-\lambda)_{\lambda+1}\frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)}\varphi(k)a_k|z|^k$$

$$\times\frac{1}{\sum_{s=1}^{m}\frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}b_{sj-(s-1)(p+\eta)}}$$

$$\leq |z|^2$$

$$\times\frac{\varphi(2)\frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)}}{\sum_{s=1}^{m}\frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}b_{sj-(s-1)(p+\eta)}}$$

$$\times\sum_{k=2}^{\infty}(k+p+\eta-\lambda)_{\lambda+1}a_k$$

$$= |z|^2$$

$$\times\frac{\frac{\Gamma(2+p+\eta-\lambda)}{\Gamma(3+p+\eta-\lambda-n)}\frac{\Gamma(p+\eta-\lambda+1-n)}{\Gamma(p+\eta+1)}}{\sum_{s=1}^{m}\frac{\Gamma(sj-(s-2)(p+\eta)+1)\Gamma(p+\eta-\lambda+1-v)}{\Gamma(p+\eta+1)\Gamma(sj-(s-2)(p+\eta)-\lambda+1-v)}b_{sj-(s-1)(p+\eta)}}$$

$$\times\sum_{k=2}^{\infty}(k+p+\eta-\lambda)_{\lambda+1}a_k \leq |z|^2 < 1,$$

by means of the hypothesis (18) of Theorem 4.

Now , we prove the following property by using the definition of f_i which given by

$$f_i(z) = z^{p+\eta}+\sum_{k=2}^{\infty}a_{k,i}\,z^{k+p+\eta}. \qquad (21)$$

Theorem 5.

Let f of the form (1) be in the class $L(p,\eta,\delta)$ such that $f(U)$ is convex. Assume that f satisfies the inequality (5). Then , for the Cesàro operator [3-6] of f defined by the relation

$$\sigma_k(z) = \sum_{k=2}^{\infty}\frac{1}{k+1}\left(\sum_{i=0}^{k}a_{k,i}\right)z^{k+p+\eta},$$

$$(p=1,2,\ldots,0\leq\eta<1\,,z\in U)$$

with $\sigma_0(z)=0,\sigma_1(z)=z^{p+\eta}$, we have $\sigma_k(z)\in L(p,\eta,\delta)$.

Proof. Since f_i which is defined by (21) satisfies (5) then in view of Theorem 1 , we have

$$\sum_{k=2}^{\infty}a_{k,i}(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]$$

$$\times\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}$$

$$\leq \gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta).$$

For all $k\in\mathbb{N}_0$, we have

$$\sigma_k(z) = 0+z^{p+\eta}$$

$$+\sum_{k=2}^{\infty}\frac{1}{k+1}\left(\sum_{i=0}^{k}a_{k,i}\right)z^{k+p+\eta}, \quad (z\in U).$$

Hence

$$\sum_{k=2}^{\infty}(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]$$

$$\times\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta}\frac{1}{k+1}\left(\sum_{i=0}^{k}a_{k,i}\right)$$

$$= \frac{1}{k+1}\sum_{i=0}^{n}(\sum_{k=2}^{\infty}(k+p+\eta)[1+\mu(k+\gamma|\beta|-2)]$$

$$\times\left(\frac{c+p+\eta}{c+k+p+\eta}\right)^{\delta} a_{k,i})$$

$$< \frac{1}{k+1}\sum_{i=0}^{k}\big(\gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta)\big)$$

$$= \gamma|\beta|(1-\mu)+(\gamma|\beta|-1)\mu(p+\eta).$$

This implies that $\sigma_k(z) \in L(p,\eta,\delta)$.

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