

ON A CERTAIN SUBCLASS OF HARMONIC UNIVALENT
FUNCTIONS BY APPLYING AN OPERATOR

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Abstract: In the present paper, we study various properties of the harmonic univalent functions with analytic nature by applying the operator $\mathcal{I}_\sigma^\delta$. We have obtained coefficient estimate for a function to be in the class $W_{\mathcal{HCV}}(k, \beta, \lambda, \delta, \sigma)$ of harmonic univalent functions. We have also obtained convolution condition, convex combination, distortion bounds and extreme points for harmonic univalent functions.

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1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D . We can write $f = h + \bar{g}$, where h and g are analytic in D .

Definition 1.1. Let $f = u + iv$ be harmonic in D , then there exist analytic functions G, H such that $u = \operatorname{Re} G$ and $v = \operatorname{Im} H$, therefore $f = u + iv = h + \bar{g}$, where $h = \frac{G+H}{2}, \bar{g} = \frac{\bar{G}-\bar{H}}{2}$ and we call h and g analytic part and co-analytic part of f respectively.

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The Jacobian of f is given by $\mathcal{J}_f(z) = |h'(z)|^2 - |g'(z)|^2$, also we show by $w(z)$ the dilatation function for f and define $w(z) = \frac{g'(z)}{h'(z)}$. Lewy [11], Clunie and Sheil-Small [3] have shown that the mapping $z \rightarrow f(z)$ is sense preserving and locally injective in D if and only if $\mathcal{J}_f > 0$ in D .

Definition 1.2. Let f be a harmonic function ($f = h + \bar{g}$), then f is said to be univalent in D if the mapping $z \rightarrow f(z)$ is sense preserving and injective in D . We denote by \mathcal{H} the class of all harmonic functions $f = h + \bar{g}$ that are univalent and sense preserving in the open unit disk $U = \{z : |z| < 1\}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1 \tag{1.1}$$

with the normalization conditions $f(0) = 0, f_z(0) = 1$, where $f_z(0)$ denote the partial derivative of $f(z)$ at $z = 0$.

In case $g \equiv 0$ this class reduces to the class T of analytic univalent functions, this class with famous subclass $S^*(\beta)$ (starlike of order β) and $C(\beta)$ (convex of order β) were studied by Robertson [13], MacGregor [12], Jack [4] and others.

Also denote by $\bar{\mathcal{H}}$ the subclass of \mathcal{H} consisting of the functions $f = h + \bar{g}$, where h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \tag{1.2}$$

We denote by $k-UCV, 0 \leq k < \infty$, the class of k -uniformly convex univalent functions and we say that $h(z) \in k-UCV$ if and only if

$$\operatorname{Re} \left\{ 1 + (1 + ke^{i\theta}) \left(\frac{zh''(z)}{h'(z)} \right) \right\} \geq \beta, \quad \text{for real } \theta$$

and $0 \leq \beta < 1$ (see Kanas and Wiśniowska [9], and Kanas and Srivastava [8]). The class \mathcal{H} with some different subclasses are investigated by many authors, for example Jahangiri [5], [6], Jahangiri and Silverman [7], Rosy et al [14], Clunie and Sheil-Small [3] and others.

Definition 1.3. The operator of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $\delta \geq 0, \sigma \geq 0$ is denoted by $\mathcal{I}_{\sigma}^{\delta}$ and defined as following:

$$\mathcal{I}_{\sigma}^{\delta} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{n + \delta}{1 + \delta} \right)^{\sigma} a_n z^n \quad (z \in U). \tag{1.3}$$

The operator $\mathcal{I}_{\sigma}^{\delta}$ was studied recently by Cho and Srivastava [2] and Cho and Kim [1], see also [16] and [17].

Also we define the operator $\mathcal{I}_\sigma^\delta$ of harmonic functions $f = h + \bar{g}$ in \mathcal{H} by

$$\mathcal{I}_\sigma^\delta f(z) = \mathcal{I}_\sigma^\delta h(z) + \overline{\mathcal{I}_\sigma^\delta g(z)}, \tag{1.4}$$

where $\mathcal{I}_\sigma^\delta f(z)$ is given by (1.3).

Definition 1.4. Let $f = h + \bar{g}$, be the harmonic function of the form (1.1), then $f \in W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$ if and only if

$$\begin{aligned} \operatorname{Re} \{ & (1 + ke^{i\theta}) [z(\mathcal{I}_\sigma^\delta h(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta h(z))'' \\ & + \overline{(2\lambda - 1)z(\mathcal{I}_\sigma^\delta g(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta g(z))''}] / [(1 - \lambda)(\mathcal{I}_\sigma^\delta h(z) \\ & + \overline{\mathcal{I}_\sigma^\delta g(z)}) + \lambda(z(\mathcal{I}_\sigma^\delta h(z))' - \overline{z(\mathcal{I}_\sigma^\delta g(z))'})] - ke^{i\theta} \} \geq \beta, \end{aligned} \tag{1.5}$$

where $0 \leq \theta \leq 2\pi, 0 \leq \lambda \leq 1, k \geq 0, 0 \leq \beta < 1, \delta \geq 0$ and $\sigma \geq 0$. We note that when $\lambda = 1, \sigma = 0$, the family $W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$ reduces to the class $W_{\mathcal{H}CV}(k, \beta)$, the class of uniformly convex analytic functions defined by [10]. Also when $k = 1, \lambda = 1$ and $\sigma = 0$, we obtain the class of harmonic convex function $GK_{\mathcal{H}}(\beta)$, which is defined by [15], when $k = 1, \lambda = 0$ and $\sigma = 0$, we get the class of harmonic starlike function $G_{\mathcal{H}}(\beta)$, studied by [14].

2. Main Results

We need the following lemma due to [6].

Lemma 2.1. Let f be of the form (1.1). If

$$\sum_{n=1}^{\infty} [(n - \beta)|a_n| + (n + \beta)|b_n|] \leq 1 - \beta,$$

then f is harmonic, orientation preserving univalent in U .

We begin by proving a sufficient condition for functions to be in $W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$.

Theorem 2.1. Let $f = h + \bar{g}$ be given by (1.1) and

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk - k - \beta)}{1 - \beta} |a_n| \\ + \sum_{n=1}^{\infty} \frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk + k + \beta)}{1 - \beta} |b_n| \leq 1, \end{aligned} \tag{2.1}$$

where $a_1 = 1, k \geq 0, 0 \leq \lambda \leq 1, 0 \leq \beta < 1, \delta \geq 0$ and $\sigma \geq 0$. Then f is harmonic univalent in U and $f \in W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$.

Proof. We have to confirm that f is locally univalent and sense-preserving

in U . Since $(1+n\lambda-\lambda) \geq 1, \frac{n+\delta}{1+\delta} \geq 1, n-\beta \leq (n+nk-k-\beta) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (1+n\lambda-\lambda)$ and $n+\beta \leq (1+n\lambda-\lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk+k+\beta)$ for $k \geq 0, 0 \leq \lambda \leq 1$, it follows from Lemma 2.1 that f is harmonic, sense-preserving, univalent.

On the other hand, it is sufficient to show that if (2.1) holds, then:

$$\begin{aligned} & \operatorname{Re} \{ (1 + ke^{i\theta}) [z(\mathcal{I}_\sigma^\delta h(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta h(z))'' \\ & \quad + \overline{(2\lambda - 1)z(\mathcal{I}_\sigma^\delta g(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta g(z))''}] / [(1 - \lambda)(\mathcal{I}_\sigma^\delta h(z) \\ & \quad + \overline{\mathcal{I}_\sigma^\delta g(z)}) + \lambda(z(\mathcal{I}_\sigma^\delta h(z))' - \overline{z(\mathcal{I}_\sigma^\delta g(z))'})] - ke^{i\theta} \} \geq \beta, \end{aligned}$$

where $z = re^{i\theta}, 0 \leq \theta \leq 2\pi, 0 \leq r < 1, k \geq 0, 0 \leq \beta < 1, 0 \leq \lambda \leq 1, \delta \geq 0$ and $\sigma \geq 0$. Let

$$\begin{aligned} E(z) = & [z(\mathcal{I}_\sigma^\delta h(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta h(z))'' \\ & + \overline{(2\lambda - 1)z(\mathcal{I}_\sigma^\delta g(z))' + \lambda z^2(\mathcal{I}_\sigma^\delta g(z))''}] (1 + ke^{i\theta}) - ke^{i\theta} [(1 - \lambda) \\ & \quad \times (\mathcal{I}_\sigma^\delta h(z) + \overline{\mathcal{I}_\sigma^\delta g(z)}) + \lambda(z(\mathcal{I}_\sigma^\delta h(z))' - \overline{z(\mathcal{I}_\sigma^\delta g(z))'})] \end{aligned}$$

and

$$F(z) = (1 - \lambda)(\mathcal{I}_\sigma^\delta h(z) + \overline{\mathcal{I}_\sigma^\delta g(z)}) + \lambda(z(\mathcal{I}_\sigma^\delta h(z))' - \overline{z(\mathcal{I}_\sigma^\delta g(z))'}).$$

Using the fact that $\operatorname{Re}(w) \geq \beta$ if and only if $|w + 1 - \beta| \geq |w - (1 + \beta)|$. We want to show that

$$|E(z) + (1 - \beta)F(z)| - |(1 + \beta)F(z) - E(z)| \geq 0. \tag{2.2}$$

Substituting for $E(z)$ and $F(z)$ in (2.2), we get

$$\begin{aligned} & |E(z) + (1 - \beta)F(z)| - |(1 + \beta)F(z) - E(z)| \\ & = |(2 - \beta)z + \sum_{n=2}^\infty (1 + n\lambda - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^\sigma [n + 1 - \beta \\ & \quad + ke^{i\theta}(n - 1)] a_n z^n + \sum_{n=1}^\infty n\lambda \left(\frac{n + \delta}{1 + \delta}\right)^\sigma [(n - 1) + \beta \\ & \quad + ke^{i\theta}(n + 1)] \bar{b}_n \bar{z}^n - \sum_{n=1}^\infty (1 - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^\sigma [n - 1 + \beta \\ & \quad + ke^{i\theta}(n + 1)] \bar{b}_n \bar{z}^n - |\beta z - \sum_{n=2}^\infty (1 + n\lambda - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^\sigma [n - 1 - \beta \\ & \quad + ke^{i\theta}(n - 1)] a_n z^n - \sum_{n=1}^\infty n\lambda \left(\frac{n + \delta}{1 + \delta}\right)^\sigma \end{aligned}$$

$$\begin{aligned}
 & \times [n + 1 + \beta + ke^{i\theta}(n + 1)]\bar{b}_n\bar{z}^n - \sum_{n=1}^{\infty} (1 - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} [n + 1 + \beta \\
 & + ke^{i\theta}(n + 1)]\bar{b}_n\bar{z}^n \geq (2 - \beta)|z| - \sum_{n=2}^{\infty} (n\lambda + 1 - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} \\
 & \times (n + nk - \beta - k - 1)|a_n||z|^n - \sum_{n=1}^{\infty} (n\lambda + 1 - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} \\
 & \times (n + nk + k + \beta - 1)|b_n||z|^n - \beta z - \sum_{n=2}^{\infty} (n\lambda + 1 - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} \\
 & \times (n + nk - k - \beta + 1)|a_n||z|^n - \sum_{n=1}^{\infty} (n\lambda + 1 - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} \\
 & \times (n + nk + k + \beta + 1)|b_n||z|^n \geq 2(1 - \beta)|z| - 2 \sum_{n=2}^{\infty} (1 + n\lambda - \lambda) \\
 & \times \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} (n + nk - k - \beta)|a_n||z|^n - 2 \sum_{n=1}^{\infty} (1 + n\lambda - \lambda) \\
 & \times \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} (n + nk + k + \beta)|b_n||z|^n \geq 0 \quad (\text{by (2.1)}).
 \end{aligned}$$

To confirm the sharpness of the theorem, we have the harmonic function

$$\begin{aligned}
 f(z) = z + \sum_{n=2}^{\infty} \frac{(1 - \beta)M_n z^n}{(1 + n\lambda - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} (n + nk - k - \beta)} \\
 + \sum_{n=1}^{\infty} \frac{(1 - \beta)N_n \bar{z}^n}{(1 + n\lambda - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} (n + nk + k + \beta)}, \quad (2.3)
 \end{aligned}$$

where $\sum_{n=2}^{\infty} |M_n| + \sum_{n=1}^{\infty} |N_n| = 1$, show that the coefficient bound given in Theorem 2.1 is sharp. Now

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \frac{(1 + n\lambda - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} (n + nk - k - \beta)}{1 - \beta} |a_n| \\
 & + \sum_{n=1}^{\infty} \frac{(1 + n\lambda - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} (n + nk + k + \beta)}{1 - \beta} |b_n| = \sum_{n=2}^{\infty} |M_n| + \sum_{n=1}^{\infty} |N_n| = 1.
 \end{aligned}$$

Therefore the function of the form (2.3) are in $W_{\mathcal{HCV}}(k, \beta, \lambda, \delta, \sigma)$. □

Theorem 2.2. *Let $f = h + \bar{g}$ with h and g of the form (1.2). Then $f \in W_{\overline{H}CV}(k, \beta, \lambda, \delta, \sigma)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} (n + nk - k - \beta)}{1 - \beta} |a_n| + \sum_{n=1}^{\infty} \frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} (n + nk + k + \beta)}{1 - \beta} |b_n| \leq 1, \tag{2.4}$$

where $a_1 = 1, k \geq 0, 0 \leq \lambda \leq 1, n \in \mathbb{N}, 0 \leq \beta < 1, \delta \geq 0$ and $\sigma \geq 0$.

Proof. Clearly, by Theorem 2.1, if $f(z)$ given by (1.2) satisfies (2.4), that is in $W_{\mathcal{H}CV}(k, \beta, \lambda, \delta, \sigma)$, then $f(z) \in W_{\overline{H}CV}(k, \beta, \lambda, \delta, \sigma)$. Conversely it is sufficient to show that $f \notin W_{\overline{H}CV}(k, \beta, \lambda, \delta, \sigma)$ if the condition (2.4) does not hold.

The necessary and sufficient condition for $f(z) = h(z) + \bar{g}(z)$ defined by (1.2), to be in $W_{\overline{H}CV}(k, \beta, \lambda, \delta, \sigma)$, is that the condition (1.3) holds true, thus we must show that

$$\begin{aligned} \operatorname{Re} \{ (1 + ke^{i\theta}) [z(\mathcal{I}_{\sigma}^{\delta} h(z))' + \lambda z^2(\mathcal{I}_{\sigma}^{\delta} h(z))'' + \overline{(2\lambda - 1)z(\mathcal{I}_{\sigma}^{\delta} g(z))' + \lambda z^2(\mathcal{I}_{\sigma}^{\delta} g(z))''}] \\ / [(1 - \lambda)(\mathcal{I}_{\sigma}^{\delta} h(z) + \overline{\mathcal{I}_{\sigma}^{\delta} g(z)}) + \lambda(z(\mathcal{I}_{\sigma}^{\delta} h(z))' - \overline{z(\mathcal{I}_{\sigma}^{\delta} g(z))'})] \\ - (ke^{i\theta} + \beta) \} = \operatorname{Re} \left\{ \frac{I(z)}{J(z)} \right\} \geq 0, \end{aligned}$$

where

$$\begin{aligned} I(z) = (1 - \beta)z - \left[\sum_{n=2}^{\infty} (1 + n\lambda - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} ((n - \beta) \right. \\ \left. + (n - 1)ke^{i\theta}) |a_n| z^n + \sum_{n=1}^{\infty} n\lambda \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} ((n + \beta) \right. \\ \left. + (n + 1)ke^{i\theta}) |b_n| \bar{z}^n - \sum_{n=1}^{\infty} (1 - \lambda)((n + \beta) + (n + 1)ke^{i\theta}) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} |b_n| \bar{z}^n \right] \end{aligned}$$

and

$$\begin{aligned} J(z) = z - \left[\sum_{n=2}^{\infty} (1 + n\lambda - \lambda) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} |a_n| z^n \right. \\ \left. + \sum_{n=1}^{\infty} (n\lambda + \lambda - 1) \left(\frac{n + \delta}{1 + \delta}\right)^{\sigma} |b_n| \bar{z}^n \right]. \end{aligned}$$

By choosing the values of z on the positive real axis, where $0 \leq z = r < 1$ and since $\operatorname{Re}(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, then we must show that

$$\begin{aligned}
 & \frac{(1 - \beta) - \sum_{n=2}^{\infty} (1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} (n + nk - k - \beta)|a_n|}{1 - \left\{ \sum_{n=2}^{\infty} (n\lambda + 1 - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} |a_n| + \sum_{n=1}^{\infty} (n\lambda - 1 + \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} |b_n| \right\}} \\
 & - \frac{\sum_{n=1}^{\infty} (1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} (n + nk + k + \beta)|b_n|}{1 - \left\{ \sum_{n=2}^{\infty} (n\lambda + 1 - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} |a_n| + \sum_{n=1}^{\infty} (n\lambda - 1 + \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} |b_n| \right\}} \\
 & \geq 0. \tag{2.5}
 \end{aligned}$$

We note that (2.5) is negative for r sufficiently close to 1 when the condition (2.4) does not hold. Therefore there exists a $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.5) is negative. Then as $f \in W_{\overline{\mathcal{H}CV}}(k, \beta, \lambda, \delta, \sigma)$ we get contradiction. This completes the proof. \square

Now, we define the convolution of two harmonic functions.

Let us have harmonic functions

$$f(z) = z - \sum_{n=2}^{\infty} a_{n,1}z^n - \sum_{n=1}^{\infty} b_{n,1}\bar{z}^n \quad \text{and} \quad g(z) = z - \sum_{n=2}^{\infty} a_{n,2}z^n - \sum_{n=1}^{\infty} b_{n,2}\bar{z}^n.$$

Then we define the convolution of $f(z)$ and $g(z)$ as

$$(f * g)(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n - \sum_{n=1}^{\infty} b_{n,1}b_{n,2}\bar{z}^n.$$

Theorem 2.3. *Let $f(z) \in W_{\overline{\mathcal{H}CV}}(k, \beta, \lambda, \delta, \sigma)$ and $g(z) \in W_{\overline{\mathcal{H}CV}}(k, \alpha, \lambda, \delta, \sigma)$, then for $0 \leq \alpha \leq \beta < 1$, we have*

$$(f * g)(z) \in W_{\overline{\mathcal{H}CV}}(k, \beta, \lambda, \delta, \sigma) \subset W_{\overline{\mathcal{H}CV}}(k, \alpha, \lambda, \delta, \sigma).$$

Proof. Since $f(z) \in W_{\overline{\mathcal{H}CV}}(k, \beta, \lambda, \delta, \sigma)$, and $g(z) \in W_{\overline{\mathcal{H}CV}}(k, \alpha, \lambda, \delta, \sigma)$, then $f(z), g(z)$ satisfy the coefficient inequality (2.4). The coefficient of $f * g$ as

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \left[\frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} (n + nk - k - \beta)}{1 - \beta} |a_{n,1}a_{n,2}| \right. \\
 & \quad \left. + \frac{(n\lambda - \lambda + 1) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} (n + nk + k + \beta)}{1 - \beta} |b_{n,1}b_{n,2}| \right] \\
 & \leq \sum_{n=1}^{\infty} \left[\frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^{\sigma} (n + nk - k - \beta)}{1 - \beta} |a_{n,1}| \right]
 \end{aligned}$$

$$\begin{aligned}
 & \left. + \frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk+k+\beta)}{1-\beta} |b_{n,1}| \right] \text{ (as } |a_{n,2}| < 1, |b_{n,2}| < 1) \\
 & \leq 2 \text{ (as } f \in W_{\overline{\mathcal{H}CV}}(k, \beta, \lambda, \delta, \sigma)).
 \end{aligned}$$

This completes the proof. □

Theorem 2.4. *The family $W_{\overline{\mathcal{H}CV}}(k, \beta, \lambda, \delta, \sigma)$ is closed under convex combination.*

Proof. Assume $f_i \in W_{\overline{\mathcal{H}CV}}(k, \beta, \lambda, \delta, \sigma) (i = 1, 2, 3, \dots)$, are defined by $f_i(z) = z - \sum_{n=2}^\infty |a_{i,n}|z^n - \sum_{n=1}^\infty |b_{i,n}|\bar{z}^n$, from (2.4), we have

$$\begin{aligned}
 \sum_{n=1}^\infty \left[\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk-k-\beta)}{1-\beta} |a_{i,n}| \right. \\
 \left. + \frac{(n\lambda-\lambda+1)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk+k+\beta)}{1-\beta} |b_{i,n}| \right] \leq 2. \tag{2.6}
 \end{aligned}$$

For $\sum_{i=1}^\infty \ell_i = 1, 0 \leq \ell_i \leq 1$, we can write the convex combination of f_i as

$$\sum_{i=1}^\infty \ell_i f_i(z) = z - \sum_{n=2}^\infty \left(\sum_{i=1}^\infty \ell_i |a_{i,n}| \right) z^n - \sum_{n=1}^\infty \left(\sum_{i=1}^\infty \ell_i |b_{i,n}| \right) \bar{z}^n$$

from (2.6) we have

$$\begin{aligned}
 & \sum_{n=1}^\infty \left[\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk-k-\beta)}{1-\beta} \sum_{i=1}^\infty \ell_i |a_{i,n}| \right. \\
 & \quad \left. + \frac{(n\lambda-\lambda+1)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk+k+\beta)}{1-\beta} \sum_{i=1}^\infty \ell_i |b_{i,n}| \right] \\
 & = \sum_{i=1}^\infty \ell_i \left\{ \sum_{n=1}^\infty \left[\frac{(1+n\lambda-\lambda)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk-k-\beta)}{1-\beta} |a_{i,n}| \right. \right. \\
 & \quad \left. \left. + \frac{(n\lambda-\lambda+1)\left(\frac{n+\delta}{1+\delta}\right)^\sigma (n+nk+k+\beta)}{1-\beta} |b_{i,n}| \right] \right\} \leq 2 \sum_{i=1}^\infty \ell_i \leq 2,
 \end{aligned}$$

then $\sum_{i=1}^\infty \ell_i f_i(z) \in W_{\overline{\mathcal{H}CV}}(k, \beta, \lambda, \delta, \sigma)$. This completes the proof. □

Theorem 2.5. *If $f \in W_{\overline{HCV}}(k, \beta, \lambda, \delta, \sigma)$, then*

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1 - \beta}{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)} - \frac{(1 + 2k + \beta)}{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)} |b_1| \right) r^2,$$

$$|f(z)| \geq (1 + |b_1|)r - \left(\frac{1 - \beta}{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)} - \frac{(1 + 2k + \beta)}{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)} |b_1| \right) r^2.$$

Proof. Assume $f \in W_{\overline{HCV}}(k, \beta, \lambda, \delta, \sigma)$, then we have

$$|f(z)| \leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^2$$

$$\leq (1 + |b_1|)r + \frac{1 - \beta}{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)}$$

$$\times \sum_{n=2}^{\infty} \left[\frac{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)}{(1 - \beta)} |a_n| + \frac{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (3k + \beta + 2)}{1 - \beta} |b_n| \right] r^2$$

$$\leq (1 + |b_1|)r + \frac{1 - \beta}{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)} \left(1 - \frac{(1 + 2k + \beta)}{1 - \beta} |b_1| \right) r^2$$

$$= (1 + |b_1|)r + \left(\frac{1 - \beta}{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)} - \frac{(1 + 2k + \beta)}{(1 + \lambda) \left(\frac{2+\delta}{1+\delta}\right)^\sigma (2 + k - \beta)} |b_1| \right) r^2.$$

The second inequality can be proved by using similar arguments. □

Now we get the extreme points of the closed convex hulls of $W_{\overline{HCV}}(k, \beta, \lambda, \delta, \sigma)$ denoted by $clco W_{\overline{HCV}}(k, \beta, \lambda, \delta, \sigma)$.

Theorem 2.6. *Let f be of the form (1.2). Then f is in $clco W_{\overline{HCV}}(k, \beta,$*

λ, δ, σ) if and only if it can be expressed of the form

$$f(z) = \sum_{n=1}^{\infty} (M_n h_n + N_n g_n), \tag{2.7}$$

where

$$h_1(z) = z, \quad h_n(z) = z - \frac{1 - \beta}{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk - k - \beta)} z^n,$$

$$g_1(z) = z, \quad g_n(z) = z - \frac{1 - \beta}{(1 + n\lambda - \lambda)(n + nk + k + \beta) \left(\frac{n+\delta}{1+\delta}\right)^\sigma} \bar{z}^n,$$

$$(n = 2, 3, \dots); \quad \sum_{n=1}^{\infty} (M_n + N_n) = 1, \quad M_n \geq 0 \quad \text{and} \quad N_n \geq 0.$$

In particular, the extreme points of $W_{\mathcal{HCV}}(k, \beta, \lambda, \delta, \sigma)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. We can write (2.7) as

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (M_n h_n + N_n g_n) = \sum_{n=1}^{\infty} (M_n + N_n) z \\ &\quad - \sum_{n=2}^{\infty} \frac{1 - \beta}{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk - k - \beta)} M_n z^n \\ &\quad - \sum_{n=1}^{\infty} \frac{1 - \beta}{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk + k + \beta)} N_n \bar{z}^n. \end{aligned}$$

Then by (2.4), we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk - k - \beta)}{1 - \beta} \\ &\quad \times \frac{1 - \beta}{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk - k - \beta)} M_n \\ &\quad + \sum_{n=1}^{\infty} \frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk + k + \beta)}{1 - \beta} \times \\ &\quad \frac{1 - \beta}{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk + k + \beta)} N_n = \sum_{n=2}^{\infty} M_n + \sum_{n=1}^{\infty} N_n = 1 - M_1 \leq 1. \end{aligned}$$

Then $f \in clco W_{\mathcal{HCV}}(k, \beta, \lambda, \delta, \sigma)$.

Conversely, assume that $f \in clco W_{\overline{HCV}}(k, \beta, \lambda, \delta, \sigma)$, then by Theorem 2.2, we have

$$|a_n| \leq \frac{1 - \beta}{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk - k - \beta)}$$

and

$$|b_n| \leq \frac{1 - \beta}{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk + k + \beta)}.$$

Set

$$M_n = \frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk - k - \beta)}{1 - \beta} |a_n|, \quad n = 2, 3, 4, \dots,$$

$$N_n = \frac{(1 + n\lambda - \lambda) \left(\frac{n+\delta}{1+\delta}\right)^\sigma (n + nk + k + \beta)}{1 - \beta} |b_n|, \quad n = 1, 2, 3, \dots,$$

$$M_1 = 1 - \sum_{n=2}^{\infty} M_n - \sum_{n=1}^{\infty} N_n, \quad M_1 \geq 0.$$

By Theorem 2.2, we have $0 \leq M_n \leq 1$ and $0 \leq N_n \leq 1$.

Finally we conclude that $f(z) = \sum_{n=1}^{\infty} (M_n h_n + N_n g_n)$. This completes the proof. □

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