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Some Geometric Properties of a Certain Subclass of Univalent Functions Defined by Differential Subordination Property

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Abstract

In this paper , we have studied a certain subclass of univalent functions defined by linear operator $\theta_{\lambda, \delta, m}^{\gamma, \alpha_1}$ by using differential subordination property. We obtain some geometric properties, like , coefficient inequality , neighborhoods of the class $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$, convolution properties and integral mean inequalities for the fractional integral for this class.

Keywords: *Univalent Function, Hypergeometric Function, Linear Operator, Differential Subordination, Convolution, Integral Mean, Fractional Integral.*

1 Introduction

Let S denote the class of functions of the form:-

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also, denoted by K the subclass of S consisting of functions of the form:-

$$f(z) = z - \sum_{n=2}^{\infty} a_n |z|^n, \quad (2)$$

which are univalent and normalized in U .

For $f \in S$, and of the form (1) and $g(z) \in S$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$,

We define the Hadamard product (or convolution)

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (3)$$

For positive real values of $\alpha_1, \dots, \alpha_\delta$ and β_1, \dots, β_m ($\beta_j \neq 0, -1, \dots, j = 1, 2, \dots, m$),

The generalized hypergeometric function ${}_8F_m(z)$ is defined by

$$\begin{aligned} {}_8F_m(z) &\equiv {}_8F_m(\alpha_1, \dots, \alpha_\delta; \beta_1, \dots, \beta_m; z) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_\delta)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \end{aligned} \quad (4)$$

($\delta \leq m + 1$; $\delta, m \in N_0 = N \cup \{0\}$; $z \in U$),

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2) \dots (a+n-1), & n \in N. \end{cases} \quad (5)$$

The notation ${}_8F_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel and Laguerre polynomial.

$$H[\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_\delta; \beta_1, \beta_2, \beta_3, \dots, \beta_m]: K \rightarrow K$$

be a linear operator defined by

$$\begin{aligned} H[\alpha_1, \dots, \alpha_\delta; \beta_1, \dots, \beta_m]f(z) &= z {}_2F_m(\alpha_1, \alpha_2, \dots, \alpha_\delta; \beta_1, \beta_2, \dots, \beta_m; z) * f(z) \\ &= z - \sum_{n=2}^{\infty} y_n(\alpha_1; \delta; m) |a_n| z^n, \end{aligned} \tag{6}$$

where,

$$y_n(\alpha_1; \delta; m) = \frac{(\alpha_1)_{n-1} \dots (\alpha_\delta)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \frac{1}{(n-1)!}. \tag{7}$$

For notational simplicity, we use shorter notation

$$H_m^\delta[\alpha_1] \text{ for } H[\alpha_1, \dots, \alpha_\delta; \beta_1, \dots, \beta_m].$$

In the sequel .It follows from (6) that

$$H_0^1[1]f(z) = f(z), H_0^1[2]f(z) = zf'(z).$$

The linear operator $H_m^\delta[\alpha_1]$ is called Dziok-Srivastava operator (see [5]) introduced by Dziok and Srivastava which was subsequently extended Dziok and Raina [4] by using the generalized hypergeometric function, recently Srivastava et. al. [13] defined the linear operator $\theta_{\lambda, \delta, m}^{\gamma, \alpha_1}$ as follows:-

$$\theta_{\lambda, \delta, m}^{0, \alpha_1} f(z) = f(z)$$

$$\begin{aligned} \theta_{\lambda, \delta, m}^{1, \alpha_1} f(z) &= (1 - \lambda)H_m^\delta[\alpha_1]f(z) + \lambda \left(H_m^\delta[\alpha_1]f(z) \right)' \\ &= \theta_{\lambda, \delta, m}^{\alpha_1} f(z), \end{aligned} \tag{8}$$

$$\theta_{\lambda, \delta, m}^{2, \alpha_1} f(z) = \theta_{\lambda, \delta, m}^{\alpha_1} \left(\theta_{\lambda, \delta, m}^{1, \alpha_1} f(z) \right), \tag{9}$$

and in general,

$$\begin{aligned} \theta_{\lambda, \delta, m}^{\gamma, \alpha_1} f(z) &= \theta_{\lambda, \delta, m}^{\alpha_1} \left(\theta_{\lambda, \delta, m}^{\gamma-1, \alpha_1} f(z) \right), (0 \leq \lambda \leq 1, \delta \leq m + 1; \delta, m \in N_0 \\ &= N \cup \{0\} ; z \in U). \end{aligned} \tag{10}$$

If the function f(z) is given by (2), then we see from (6),(7),(8) and (10) that

$$\theta_{\lambda, \delta, m}^{\gamma, \alpha_1} f(z) = z - \sum_{n=2}^{\infty} y_n^\gamma(\alpha_1; \lambda; \delta; m) |a_n| z^n, \tag{11}$$

where

$$y_n^\gamma(\alpha_1; \lambda; \delta; m) = \left(\frac{(\alpha_1)_{n-1} \dots (\alpha_\delta)_{n-1} [1 + \lambda(n-1)]^\gamma}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1} (n-1)!} \right)^\gamma, \quad (12)$$

$(n \in N \setminus \{1\}, \gamma \in N_0)$.

Unless otherwise stated. We note that when $\gamma = 1$ and $\lambda = 0$, the linear operator $\theta_{\lambda, \delta, m}^{\gamma, \alpha_1}$ would reduce to the familiar Dziok-Srivastava linear operator given by (see [5]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [3], Owa [9] and Ruscheweyh [10].

For two analytic functions $f, g \in K$, we say that f is subordinate to g written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$, $z \in U$.

Furthermore, if the function $g(z)$ is univalent in U , then we have the following equivalence (see [8]):

$$f(z) \prec g(z) \leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition (1): For any function $f \in K$ and $\phi \geq 0$, the ϕ -neighborhood f is defined as,

$$N_{n, \phi}(f) = \{g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n \in K; \sum_{n=2}^{\infty} n ||a_n| - |b_n|| \leq \phi\}. \quad (13)$$

In particular, for the function $e(z) = z$, we see that,

$$N_{n, \phi}(e) = \{g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n \in K; \sum_{n=2}^{\infty} n |b_n| \leq \phi\}. \quad (14)$$

The concept of neighborhoods was first introduced by Goodman [6] and then generalized by Ruscheweyh [11], and studied by some authors, like Atshan [1] and Atshan and Kulkarni [2].

Definition (2): For fixed parameters A and B , with $-1 \leq B < A \leq 1$, we say that $f \in K$ is in class $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$, if it satisfies the following subordination condition:

$$1 + \frac{z(\theta_{\lambda, \delta, m}^{\gamma + \eta, \alpha_1} f(z))''}{(\theta_{\lambda, \delta, m}^{\gamma, \alpha_1} f(z))'} \prec \frac{1 + Az}{1 + Bz}. \quad (15)$$

In view of the definition of subordination, (15) is equivalent to the following condition:

$$\left| \frac{\frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'}}{B \frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'} + (B-A)} \right| < 1, \quad (z \in U).$$

For convenience, we write

$$K(\gamma, \eta, \alpha_1, \lambda, \delta, m, 1 - 2\mu, -1) = K(\gamma, \eta, \alpha_1, \lambda, \delta, m, \mu),$$

where $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, \mu)$ denotes the class of functions in K satisfying the inequality:

$$\operatorname{Re} \left\{ 1 + \frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'} \right\} > \mu, \quad 0 \leq \mu < 1; z \in U.$$

2 Neighborhoods for the Class $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$

The following theorem gives a necessary and sufficient condition for a function f to be in the class $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$.

Theorem (1): A function $f \in K$ belong to the class $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ if and only if

$$\sum_{n=2}^{\infty} n(2n-1) y_n^\gamma(\alpha_1; \lambda; \delta; m) \left((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B) \right) |a_n| \leq (A-B), \quad (16)$$

for $\gamma, \eta, m \in N_0, \delta \leq m+1, \lambda \geq 0$ and $-1 \leq B < A \leq 1$.

Proof: Let $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$. Then

$$1 + \frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'} < \frac{1+Az}{1+Bz}, \quad z \in U. \quad (17)$$

Therefore, there exists an analytic function w such that

$$w(z) = \frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{Bz(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))'' + (B-A)(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'}. \quad (18)$$

Hence,

$$|w(z)| = \left| \frac{z(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))''}{Bz(\theta_{\lambda,\delta,m}^{\gamma+\eta,\alpha_1} f(z))'' + (B-A)(\theta_{\lambda,\delta,m}^{\gamma,\alpha_1} f(z))'} \right|$$

$$= \left| \frac{-\sum_{n=2}^{\infty} n(n-1) y_n^{\gamma+\eta}(\alpha_1; \lambda; \delta; m) |a_n| z^n}{(B-A)z - \sum_{n=2}^{\infty} n^2 y_n^{\gamma}(\alpha_1; \lambda; \delta; m) (B y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (B-A)) |a_n| z^n} \right|$$

$$< 1.$$

Thus,

$$\left| \frac{\sum_{n=2}^{\infty} n(n-1) y_n^{\gamma+\eta}(\alpha_1; \lambda; \delta; m) |a_n| z^n}{(A-B)z + \sum_{n=2}^{\infty} n^2 y_n^{\gamma}(\alpha_1; \lambda; \delta; m) (B y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (B-A)) |a_n| z^n} \right| < 1.$$

Therefore,

$$Re \left\{ \frac{\sum_{n=2}^{\infty} n(n-1) y_n^{\gamma+\eta}(\alpha_1; \lambda; \delta; m) |a_n| z^n}{(A-B)z + \sum_{n=2}^{\infty} n^2 y_n^{\gamma}(\alpha_1; \lambda; \delta; m) (B y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (B-A)) |a_n| z^n} \right\} < 1. \quad (19)$$

Taking $|z|=r$, for sufficiently small r with $0 < r < 1$, since $w(z)$ is analytic for $|z|=1$. Then, the inequality (19) yields

$$\sum_{n=2}^{\infty} n(n-1) y_n^{\gamma+\eta}(\alpha_1; \lambda; \delta; m) |a_n| r^n$$

$$< (A-B)r + \sum_{n=2}^{\infty} n^2 y_n^{\gamma}(\alpha_1; \lambda; \delta; m) (B y_n^{\eta}(\alpha_1; \lambda; \delta; m) - (A-B)) |a_n| r^n.$$

Equivalently,

$$\sum_{n=2}^{\infty} n(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B)) |a_n| r^n$$

$$\leq (A-B)r$$

and (16) follows upon letting $r \rightarrow 1$.

Conversely, for $|z|=r$, $0 < r < 1$, we have $r^n < r$. That is,

$$\sum_{n=2}^{\infty} n(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B)) |a_n| r^n$$

$$\leq \sum_{n=2}^{\infty} n(2n-1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1-B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B)) |a_n| r^n$$

$$\leq (A-B)r.$$

From (16), we have

$$\left| \sum_{n=2}^{\infty} n(n-1) y_n^{\gamma+\eta}(\alpha_1; \lambda; \delta; m) |a_n| z^n \right|$$

$$\leq \sum_{n=2}^{\infty} n(n-1) y_n^{\gamma+\eta}(\alpha_1; \lambda; \delta; m) |a_n| r^n$$

$$< (A-B)r + \sum_{n=2}^{\infty} n^2 y_n^\gamma(\alpha_1; \lambda; \delta; m) (B y_n^\eta(\alpha_1; \lambda; \delta; m) + (B-A)) |a_n| r^n$$

$$< |(A-B)z| + \sum_{n=2}^{\infty} n^2 y_n^\gamma(\alpha_1; \lambda; \delta; m) (B y_n^\eta(\alpha_1; \lambda; \delta; m) + (B-A)) |a_n| z^n.$$

This proves that

$$1 + \frac{z(\theta_{\lambda, \delta, m}^{\gamma+\eta, \alpha_1} f(z))''}{(\theta_{\lambda, \delta, m}^{\gamma, \alpha_1} f(z))'} < \frac{1 + Az}{1 + Bz}, \quad z \in U,$$

and hence $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$.

Theorem (2): *If*

$$\phi = \frac{A-B}{3 \left(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1} (1+\lambda) \right)^\gamma \left[(1-B) \frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1} (1+\lambda)^\eta + (A-B) \right]}, \quad (20)$$

then $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B) \subset N_{n, \phi}(e)$.

Proof: It follows from (16), that if $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$, then

$$3 y_2^\gamma(\alpha_1; \lambda; \delta; m) \left((1-B) y_2^\eta(\alpha_1; \lambda; \delta; m) + (A-B) \right) \sum_{n=2}^{\infty} n |a_n| \leq A-B.$$

Hence

$$3 \frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1} (1+\lambda)^\gamma \left[(1-B) \left(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1} (1+\lambda) \right)^\eta + (A-B) \right]$$

$$\times \sum_{n=2}^{\infty} n|a_n| \leq A - B, \quad (21)$$

which implies that

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{A - B}{3 \left(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1} (1 + \lambda) \right)^\gamma \left[(1 - B) \left(\frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1} (1 + \lambda) \right)^\eta + (A - B) \right]} = \emptyset \quad (22)$$

Using (14), we get the result.

Definition (3): The function g defined by $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$ is said to be a member of the class $K_c(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ if there exist a function $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ such that

$$\left| \frac{g(z)}{f(z)} - 1 \right| \leq 1 - c, \quad (z \in U, 0 \leq c < 1). \quad (23)$$

Theorem (3): If $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ and

$$c = 1 - \frac{3\emptyset y_2^\gamma(\alpha_1; \lambda; \delta; m)((1 - B) y_2^\gamma(\alpha_1; \lambda; \delta; m) + (A - B))}{6 y_2^\gamma(\alpha_1; \lambda; \delta; m)((1 - B) y_2^\gamma(\alpha_1; \lambda; \delta; m) + (A - B)) - (A - B)}, \quad (24)$$

then $N_{n,\emptyset}(f) \subset K_c(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$.

Proof: Let $g \in N_{n,\emptyset}(f)$. Then we have from (13) that

$$\sum_{n=2}^{\infty} n||a_n| - |b_n|| \leq \emptyset,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} ||a_n| - |b_n|| \leq \frac{\emptyset}{2}.$$

Also, since $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$, we have from (16)

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{A - B}{6 y_2^\gamma(\alpha_1, \lambda, \delta, m)((1 - B) y_2^\eta(\alpha_1; \lambda; \delta; m) + (A - B))},$$

where

$$y_2^\gamma(\alpha_1; \lambda; \delta; m) = \frac{(\alpha_1)_1 \dots (\alpha_\delta)_1}{(\beta_1)_1 \dots (\beta_m)_1} (1 + \lambda)^\gamma,$$

$$y_2^\eta(\alpha_1; \lambda; \delta; m) = \frac{(\alpha_1)_1, \dots, (\alpha_\delta)_1}{(\beta_1)_1, \dots, (\beta_m)_1} (1 + \lambda)^\eta,$$

so that

$$\left| \frac{g(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{n=2}^\infty (|a_n| - |b_n|)z^n}{z - \sum_{n=2}^\infty |a_n|z^n} \right| < \frac{\sum_{n=2}^\infty ||a_n| - |b_n||}{1 - \sum_{n=2}^\infty |a_n|}$$

$$\leq \phi \cdot \frac{3 y_2^\gamma(\alpha_1; \lambda; \delta; m)((1 - B) y_2^\eta(\alpha_1; \lambda; \delta; m) + (A - B))}{6 y_2^\gamma(\alpha_1; \lambda; \delta; m)((1 - B) y_2^\eta(\alpha_1; \lambda; \delta; m) + (A - B)) - (A - B)}$$

=1-c.

Thus by Definition (3), $g \in K_c(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ for c given by (24). This completes the proof.

3 Convolution Properties

Theorem (4): Let the functions $f_j (j=1,2)$ defined by

$$f_j(z) = z - \sum_{n=2}^\infty |a_{n,j}|z^n, (j = 1,2), \tag{25}$$

be in the class $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$. Then $f_1 * f_2 \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, \sigma)$, where

$$\sigma \leq \frac{An(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m)((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))^2 - (A - B)^2(A + y_n^\eta(\alpha_1; \lambda; \delta; m))}{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m)((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))^2 - (A - B)^2(1 + y_n^\eta(\alpha_1; \lambda; \delta; m))}$$

Proof: We must find the largest σ such that

$$\sum_{n=2}^\infty \frac{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m)((1 - \sigma) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - \sigma))}{A - \sigma} |a_{n,1}| |a_{n,2}| \leq 1.$$

Since $f_j \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B) (j = 1,2)$, then

$$\sum_{n=2}^\infty \frac{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m)((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))}{A - B} |a_{n,j}| \leq 1, (j = 1,2). \tag{26}$$

By Cauchy –Schwarz inequality, we get

$$\sum_{n=2}^{\infty} \frac{n(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B))}{A-B} \sqrt{|a_{n,1}| |a_{n,2}|} \leq 1. \quad (27)$$

We want only to show that

$$\frac{n(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-\sigma) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-\sigma))}{A-\sigma} |a_{n,1}| |a_{n,2}| \leq$$

$$\frac{n(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B))}{A-B} \sqrt{|a_{n,1}| |a_{n,2}|}.$$

This equivalently to

$$\sqrt{|a_{n,1}| |a_{n,2}|} \leq \frac{(A-\sigma) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B))}{(A-B) ((1-\sigma) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-\sigma))}.$$

From (27), we have

$$\sqrt{|a_{n,1}| |a_{n,2}|} \leq \frac{A-B}{n(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B))}.$$

Thus, it is sufficient to show that

$$\frac{A-B}{n(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B))} \leq$$

$$\frac{(A-\sigma) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B))}{(A-B) ((1-\sigma) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-\sigma))},$$

which implies to

$$\sigma \leq \frac{An(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B))^2 - (A-B)^2 (A + y_n^{\eta}(\alpha_1; \lambda; \delta; m))}{n(2n-1) y_n^{\gamma}(\alpha_1; \lambda; \delta; m) ((1-B) y_n^{\eta}(\alpha_1; \lambda; \delta; m) + (A-B))^2 - (A-B)^2 (1 + y_n^{\eta}(\alpha_1; \lambda; \delta; m))}$$

This completes the proof.

Theorem (5): Let the functions f_j ($j=1, 2$) defined by (25) be in the class $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$. Then the function h defined by

$$h(z) = z - \sum_{n=2}^{\infty} (|a_{n,1}|^2 + |a_{n,2}|^2) z^n \quad (28)$$

belong to the class $K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, \epsilon)$, where

$$\epsilon \leq \frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))^2 - 2y_n^\eta(\alpha_1; \lambda; \delta; m)(A-B)^2 - 2(A-B)^2}{An(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))^2 + 2A(A-B)^2 + 2y_n^\eta(\alpha_1; \lambda; \delta; m)(A-B)^2}$$

Proof: We must find the largest ϵ such that

$$\sum_{n=2}^{\infty} \frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-\epsilon)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-\epsilon))}{A-\epsilon} (|a_{n,1}|^2 + |a_{n,2}|^2) \leq 1.$$

Since $f_j \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ ($j = 1, 2$), we get

$$\begin{aligned} & \sum_{n=2}^{\infty} \left(\frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))}{A-B} \right)^2 |a_{n,2}|^2 \\ & \leq \left(\sum_{n=2}^{\infty} \frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))}{A-B} |a_{n,1}| \right)^2 \leq 1, \end{aligned} \quad (29)$$

and

$$\begin{aligned} & \sum_{n=2}^{\infty} \left(\frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))}{A-B} \right)^2 |a_{n,2}|^2 \\ & \leq \left(\sum_{n=2}^{\infty} \frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))}{A-B} |a_{n,2}| \right)^2 \leq 1. \end{aligned} \quad (30)$$

Combining the inequalities (29) and (30), gives

$$\sum_{n=2}^{\infty} \frac{1}{2} \left(\frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))}{A-B} \right)^2 (|a_{n,1}|^2 + |a_{n,2}|^2) \leq 1. \quad (31)$$

But $h \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, \epsilon)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-\epsilon)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-\epsilon))}{A-\epsilon} (|a_{n,1}|^2 + |a_{n,2}|^2) \leq 1. \quad (32)$$

The inequality (32) will be satisfied if

$$\begin{aligned} & \frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-\epsilon)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-\epsilon))}{A-\epsilon} \\ & \leq \frac{(n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))^2}{2(A-B)^2}, (n = 2, 3, \dots), \end{aligned} \quad (33)$$

so that,

$$\epsilon \leq \frac{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))^2 - 2y_n^\gamma(\alpha_1; \lambda; \delta; m)(A-B)^2 - 2(A-B)^2}{An(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))^2 + 2A(A-B)^2 + 2y_n^\gamma(\alpha_1; \lambda; \delta; m)(A-B)^2}$$

This completes the proof.

4 Integral Mean Inequalities for the Fractional Integral

Definition (4) [12]: The fractional integral of order s ($s > 0$) is defined for a function f by:

$$D_z^{-s} f(z) = \frac{1}{\Gamma(s)} \int_0^z \frac{f(t)}{(z-t)^{1-s}} dt,$$

where the function f is an analytic in a simply-connected region of the complex z -plane containing the origin, and multiplicity of $(z-t)^{s-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

In 1925, Littlewood [7] proved the following subordination theorem:-

Theorem (6) (Littlewood [7]): If f and g are analytic in U with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$)

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta$$

Theorem (7): Let $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$ and suppose that f_n is defined by

$$f_n(z) = z - \frac{A-B}{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))} z^n, (n \geq 2). \quad (34)$$

Also, let

$$\sum_{n=2}^{\infty} (i-\tau)_{\tau+1} |a_i| \leq \frac{(A-B)\Gamma(n+1)\Gamma(s+\tau+3)}{n(2n-1)y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B)y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B))\Gamma(n+s+\tau+1)\Gamma(2-\tau)}, \quad (35)$$

for $0 \leq \tau \leq i, s > 0$, where $(i-\tau)_{\tau+1}$ denote the Pochhammer symbol defined by $(i-\tau)_{\tau+1} = (i-\tau)(i-\tau+1) \dots i$.

If there exists an analytic function q defined by

$$(q(z))^{n-1} = \frac{n(2n-1) y_n^\gamma(\alpha_1; \lambda; \delta; m)(1-B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B)\Gamma(n+s+\tau+1)}{(A-B)\Gamma(n+1)} \times \sum_{i=2}^{\infty} (i-\tau)_{\tau+1} H(i) |a_i| z^{i-1} \quad , \quad (36)$$

where $i \geq \tau$ and

$$H(i) = \frac{\Gamma(i-\tau)}{\Gamma(i+s+\tau+1)}, \quad (s > 0, i \geq 2), \quad (37)$$

then ,for $z = re^{i\phi}$ and $0 < r < 1$

$$\int_0^{2\pi} |D_z^{-s-\tau} f(z)|^\mu d\phi \leq \int_0^{2\pi} |D_z^{-s-\tau} f_n(z)|^\mu d\phi \quad (s > 0, \mu > 0). \quad (38)$$

Proof: Let $f(z) = z - \sum_{i=2}^{\infty} |a_i| z^i$. For $\tau \geq 0$ and Definition (4), we get

$$\begin{aligned} D_z^{-s-\tau} f(z) &= \frac{\Gamma(2)z^{s+\tau+1}}{\Gamma(s+\tau+2)} \left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(i+1)\Gamma(s+\tau+2)}{\Gamma(2)\Gamma(i+s+\tau+1)} |a_i| z^{i-1}\right) \\ &= \frac{\Gamma(2)z^{s+\tau+1}}{\Gamma(s+\tau+2)} \left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\tau+2)}{\Gamma(2)} (i-\tau)_{\tau+1} H(i) |a_i| z^{i-1}\right), \end{aligned}$$

where

$$H(i) = \frac{\Gamma(i-\tau)}{\Gamma(i+s+\tau+1)}, \quad (s > 0, i \geq 2)$$

Since H is a decreasing function of i , we have

$$0 < H(i) \leq H(2) = \frac{\Gamma(2-\tau)}{\Gamma(s+\tau+3)}.$$

Similarly, from (34) and Definition (4), we get

$$D_z^{-s-\tau} f_n(z) = \frac{\Gamma(2)z^{s+\tau+1}}{\Gamma(s+\tau+2)} \left(1 - \frac{(A-B)\Gamma(n+1)\Gamma(s+\tau+2)}{n(2n-1) y_n^\gamma(\alpha_1; \lambda; \delta; m)((1-B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B)\Gamma(n+s+\tau+1)} z^{n-1}\right).$$

For $\mu > 0$, and $z = re^{i\phi}$ ($0 < r < 1$), we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \tau + 2)}{\Gamma(2)} (i - \tau)_{\tau+1} H(i) |a_i| z^{1-i} \right| d\varphi \leq \int_0^{2\pi} \left| 1 - \frac{(A - B)\Gamma(n + 1)\Gamma(s + \tau + 2)}{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))\Gamma(2)\Gamma(n + s + \tau + 1)} z^{n-1} \right|^\mu d\varphi.$$

By applying Littlewood’s subordination theorem, it would suffice to show that

$$1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \tau + 2)}{\Gamma(2)} (i - \tau)_{\tau+1} H(i) |a_i| z^{i-1} < 1 - \frac{(A - B)\Gamma(n + 1)\Gamma(s + \tau + 2)}{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))\Gamma(2)\Gamma(n + s + \tau + 1)} z^{n-1}.$$

By setting

$$1 - \sum_{i=2}^{\infty} \frac{\Gamma(s + \tau + 2)}{\Gamma(2)} (i - \tau)_{\tau+1} H(i) |a_i| z^{i-1} = \frac{(A - B)\Gamma(n + 1)\Gamma(s + \tau + 2)}{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))\Gamma(2)\Gamma(n + s + \tau + 1)} (q(z))^{n-1},$$

we find that

$$(q(z))^{n-1} = \frac{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))\Gamma(n + s + \tau + 1)}{(A - B)\Gamma(n + 1)} \times \sum_{i=2}^{\infty} (i - \tau)_{\tau+1} H(i) |a_i| z^{i-1},$$

which readily yields $w(0)=0$. For such a function q , we obtain

$$(q(z))^{n-1} \leq \frac{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))\Gamma(n + s + \tau + 1)}{(A - B)\Gamma(n + 1)} \times \sum_{i=2}^{\infty} (i - \tau)_{\tau+1} H(i) |a_i| |z|^{i-1} \leq \frac{n(2n - 1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1 - B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A - B))\Gamma(n + s + \tau + 1)}{(A - B)\Gamma(n + 1)} \times H(2) |z| \sum_{i=2}^{\infty} (i - \tau)_{\tau+1} |a_i|$$

$$= |z| \frac{n(2n-1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1-B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B)) \Gamma(n+s+\tau+1)}{(A-B) \Gamma(s+\tau+3) \Gamma(n+1)}$$

$$\times \Gamma(2-\tau) \sum_{i=2}^{\infty} (i-\tau)_{\tau+1} |a_i| \leq |z| < 1.$$

This completes the proof.

By taking $\tau = 0$, in the Theorem 7, we have the following corollary:

Corollary (1): Let $f \in K(\gamma, \eta, \alpha_1, \lambda, \delta, m, A, B)$, and suppose that f_n is defined by (34). Also let

$$\sum_{i=2}^{\infty} i |a_i| \leq \frac{(A-B) \Gamma(n+1) \Gamma(s+3)}{n(2n-1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1-B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B)) \Gamma(2) \Gamma(n+s+1)}$$

$n \geq 2$. If there exists an analytic function q defined by

$$(q(z))^{n-1} = \frac{n(2n-1) y_n^\gamma(\alpha_1; \lambda; \delta; m) ((1-B) y_n^\eta(\alpha_1; \lambda; \delta; m) + (A-B)) \Gamma(n+s+1)}{(A-B) \Gamma(n+1)}$$

$$\times \sum_{i=2}^{\infty} i H(i) |a_i| z^{i-1} ,$$

where

$$H(i) = \frac{\Gamma(i)}{\Gamma(i+s+1)} , (s > 0, i \geq 2).$$

Then for $z = r e^{i\varphi}$ and $0 < r < 1$

$$\int_0^{2\pi} |D_z^{-s} f(z)|^\mu d\varphi \leq \int_0^{2\pi} |D_z^{-s} f_n(z)|^\mu d\theta \quad (s > 0, \mu > 0) .$$

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