

On Essentially Normality of the Composition Operator C_σ

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Abstract

In this paper we have studied the composition operator induced by the automorphism σ and discussed the adjoint of the composition of the symbol σ . We have look also at some known properties on composition operators and tried to see the analogue properties in order to show how the results are changed by changing the function σ on U .

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties .

Introduction

In this paper we are going to the composition operator C_σ induced by the symbol σ and properties of C_σ and also discuss the adjoint of Composition Operator C_σ induced by the symbol σ and we discuss the normality of C_σ . Moreover, we study the essential normality of C_σ .

Definition(1.1) : [4]

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ is called unit ball in complex numbers $\mathbb{C}$ and $\partial U = \{z \in \mathbb{C} : |z| = 1\}$ is called boundary of U

Definition (1.2):

For $\gamma \in U$, define $\sigma(z) = \frac{z}{-1+\gamma z}$ ($z \in U$). Since the denominator equal zero only at $z = \frac{1}{\gamma}$, the function σ is holomorphic on the ball $\left\{|z| < \frac{1}{|\gamma|}\right\}$. Since $(\gamma \in U)$ then this ball contains U . Hence σ take U into U and holomorphic on U .

Definition (1.3): [4]

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n) z^n$ holomorphic on U such that $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ with $f^{\wedge}(n)$ denotes then the Taylor coefficient of f .

Remark (1.4) : [1]

We can define an inner product of the Hardy space H^2 as follows:

$f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n) z^n$ and $g(z) = \sum_{n=0}^{\infty} g^{\wedge}(n) z^n$, then inner product of f and g is:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} f^{\wedge}(n) \overline{g^{\wedge}(n)}$$

Example (1.5) :[10]

Let $K_{\alpha}(z) = \frac{1}{1-\bar{\alpha}z}$. Since $\alpha \in U$, then $|\alpha| < 1$, hence the geometric series $\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and thus $K_{\alpha} \in H^2$ and $K_{\alpha}(z) = \sum_{n=0}^{\infty} (\bar{\alpha})^n z^n$.

Definition(1.6) : [4]

Let $\psi : U \rightarrow U$ and holomorphic on U , the composition operator C_{ψ} induced by ψ is defined on H^2 by the equation $C_{\psi} f = f \circ \psi$ ($f \in H^2$)

Definition(1.7) : [2]

Let T be a bounded operator on a Hilbert space H , then the norm of an operator T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Littlewood's Subordination principle (1.8) : [11]

Let $\psi : U \rightarrow U$ and holomorphic on U with $\psi(0) = 0$, then for each $f \in H^2$, $f \circ \psi \in H^2$ and $\|f \circ \psi\| \leq \|f\|$.

Remark (1.9) : [4]

- 1) One can easily show that $C_\phi C_\psi = C_{\psi \circ \phi}$ and hence $C_\phi^n = C_\phi C_\phi \dots C_\phi = C_{\phi \circ \phi \dots \phi} = C_{\phi_n}$
- 2) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U .
- 3) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Definition(1.10): [3]

Let T be an operator on a Hilbert space H , The operator T^* is the adjoint of T if $\langle T^*x, y \rangle = \langle x, Ty \rangle$ for each $x, y \in H$.

Theorem (1.11) : [5]

$\bigcup_{\alpha \in U} \{K_\alpha\}$ forms a dense subset of H^2 .

Theorem (1.12) : [10]

Let $\psi : U \rightarrow U$ and holomorphic on U , then for all $\alpha \in U$

$$C_\psi^* K_\alpha = K_{\psi(\alpha)}$$

Definition(1.13): [11]

Let H^∞ be the set of all bounded holomorphic on U .

Definition(1.14): [6]

Let $g \in H^\infty$, the Toeplitz operator T_g is the operator on H^2 is given by :

$$(T_g f)(z) = g(z)f(z) \quad (f \in H^2, z \in U)$$

Remark (1.15) : [7]

For each $f \in H^2$, it is well- known that $T_h^* f = T_{\bar{h}} f$ such that $h \in H^\infty$.

Proposition(1.16) :

Let $\alpha \in U, C_{\sigma}^* = T_g C_{\beta} T_h^*$, where $h(z) = 1 - \gamma z, g(z) = 1, \beta(z) = \bar{\gamma} - z$

Proof :

By (1.15), $T_h^* f = T_{\bar{h}} f$ for each $f \in H^2$. Hence for all $\alpha \in U$,

$$\langle T_h^* f, K_{\alpha} \rangle = \langle T_{\bar{h}} f, K_{\alpha} \rangle = \langle f, T_{\bar{h}}^* K_{\alpha} \rangle \dots (1-1)$$

On the other hand ,

$$\langle T_h^* f, K_{\alpha} \rangle = \langle f, T_h K_{\alpha} \rangle = \langle f, h(\alpha) K_{\alpha} \rangle \dots (1-2)$$

From (1-1) and (1-2) one can see that $T_{\bar{h}}^* K_{\alpha} = h(\alpha) K_{\alpha}$. Hence $T_h^* K_{\alpha} = \overline{h(\alpha)} K_{\alpha}$.

Calculation give:

$$\begin{aligned} C_{\sigma}^* K_{\alpha}(z) &= K_{\sigma(\alpha)}(z) \\ &= \frac{1}{1 - \sigma(\alpha)z} = \frac{1}{1 - \frac{\bar{\alpha}z}{-1 + \bar{\gamma}\bar{\alpha}}} = \frac{1}{\frac{-1 + \bar{\gamma}\bar{\alpha} - \bar{\alpha}z}{-1 + \bar{\gamma}\bar{\alpha}}} \\ &= \frac{1}{\frac{-1 + \bar{\gamma}\bar{\alpha} - \bar{\alpha}z}{-1 + \bar{\gamma}\bar{\alpha}}} = \frac{1 - \bar{\gamma}\bar{\alpha}}{1 - \bar{\alpha}(\bar{\gamma} - z)} = \frac{\overline{(1 - \gamma\alpha)}}{1 - \bar{\alpha}(\bar{\gamma} - z)} \\ &= \overline{(1 - \gamma\alpha)} \cdot \frac{1}{1 - \bar{\alpha}(\bar{\gamma} - z)} \\ &= \overline{h(\alpha)} T_g K_{\alpha}(\beta(z)) = \overline{h(\alpha)} T_g C_{\beta} K_{\alpha}(z) \\ &= T_g \overline{h(\alpha)} C_{\beta} K_{\alpha}(z) = T_g C_{\beta} \overline{h(\alpha)} K_{\alpha}(z) \\ &= T_g C_{\beta} T_h^* K_{\alpha}(z), \text{ therefore} \end{aligned}$$

$$C_{\sigma}^* K_{\alpha}(z) = T_g C_{\beta} T_h^* K_{\alpha}(z).$$

But $\overline{\bigcup_{\alpha \in U} \{K_{\alpha}\}} = H^2$ then $C_{\sigma}^* = T_g C_{\beta} T_h^*$.

Definition (1.17) : [3]

Let T be an operator on a Hilbert space H , T is called normal operator if $TT^* = T^*T$ and T is called unitary operator if $TT^* = T^*T = I$, and T is called isometric operator if $T^*T = I$

Theorem (1.18) : [9]

If $\phi : U \rightarrow U$ is holomorphic map on U , then C_{ϕ} is normal if and only if $\phi(z) = \lambda z$ for some $\lambda, |\lambda| \leq 1$.

Theorem (1.19) :

If $\phi : U \rightarrow U$ be holomorphic map on U , then C_ϕ is unitary if and only if $\phi(z) = \lambda z$ for some $\lambda, |\lambda| = 1$

Proof :

Suppose C_ϕ is unitary, hence by (1.17) $C_\phi C_\phi^* = C_\phi^* C_\phi = I$, hence $C_\phi C_\phi^* = C_\phi^* C_\phi$, hence C_ϕ is normal operator, hence by (1.18) $\phi(z) = \lambda z$ for some $\lambda, |\lambda| \leq 1$. It is enough to show that $|\lambda| = 1$

$$C_\phi^* C_\phi K_\beta(z) = C_\phi^* K_\beta(\phi(z)) = K_{\phi(\beta)}(\phi(z)) .$$

$$= \frac{1}{1-\overline{\phi(\alpha)}\phi(z)} = \frac{1}{1-\overline{\lambda}\bar{\beta}\lambda z} = \frac{1}{1-|\lambda|^2\bar{\beta}z}$$

On the other hand $C_\phi^* C_\phi K_\beta(z) = K_\beta(z)$, hence $\frac{1}{1-|\lambda|^2\bar{\beta}z} = K_\beta(z) = \frac{1}{1-\bar{\beta}z}$.

Thus $|\lambda|^2\bar{\beta} = \bar{\beta}$, then $|\lambda| = 1$.

Conversely, Suppose $\phi(z) = \lambda z$ for some $\lambda, |\lambda| = 1$. For $\beta \in U$, for every $z \in U$

$$C_\phi^* C_\phi K_\beta(z) = C_\phi^* K_\beta(\phi(z)) = K_{\phi(\beta)}(\phi(z)) .$$

$$= \frac{1}{1-\overline{\phi(\beta)}\phi(z)} = \frac{1}{1-\overline{\lambda}\bar{\beta}\lambda z} = \frac{1}{1-|\lambda|^2\bar{\beta}z}$$

$$= \frac{1}{1-\bar{\beta}z} = K_\beta(z)$$

Moreover for every $z \in U$

$$C_\phi C_\phi^* K_\beta(z) = C_\phi K_{\phi(\beta)}(\phi(z)) = K_{\phi(\beta)}(\phi(z)) .$$

$$= \frac{1}{1-\overline{\phi(\beta)}\phi(z)} = \frac{1}{1-\overline{\lambda}\bar{\beta}\lambda z} = \frac{1}{1-|\lambda|^2\bar{\beta}z} = \frac{1}{1-\bar{\beta}z} = K_\beta(z)$$

hence $C_\phi C_\phi^* = C_\phi^* C_\phi = I$ on the family $V_{\alpha \in U}\{K_\alpha\}$. But by (1.11) $V_{\alpha \in U}\{K_\alpha\}$ forms a dense subset of H^2 , hence $C_\phi C_\phi^* = C_\phi^* C_\phi = I$ on H^2 . Therefore C_ϕ is unitary composition operator on H^2 .

Proposition(1.20) :

If $\gamma = 0$, then C_σ is an unitary composition operator .

Proof :

Since $\sigma(z) = \frac{z}{-1+\gamma z}$ since $\gamma = 0$, $\sigma(z) = \frac{z}{-1+\gamma z} = -z = \lambda z, \lambda = -1, |\lambda| = 1$, hence by (1.19) C_σ is unitary composition

Remark(1.21) :

From Definition (1.17), we note every unitary composition operator is a normal composition operator.

Proposition(1.22) :

If $\gamma = 0$, then C_σ is a normal composition operator .

Proof:

Since $\gamma = 0$, then C_σ is an unitary composition operator by (1.20) , hence by (1.21) C_σ is a normal composition operator

Definition (1.23) : [7]

Let T be an operator on a Hilbert space H is subnormal if there exists a normal operator S on a Hilbert space K such that H is a subspace of K , the subspace H is invariant under the operator S and the restriction of S to H coincides with T , and every normal operator is subnormal operator.

Proposition(1.24) :

If $\gamma = 0$, then C_σ is a subnormal composition operator .

Proof:

If $\gamma = 0$, then C_σ is a normal composition operator by (1.22), by (1.23) C_σ is a subnormal composition operator.

Definition (1.25) : [13]

Let T be an operator on a Hilbert space H , T is called compact if every sequence x_n in H is weakly converges to x in H , then Tx_n is strongly converges to Tx .Moreover ($x_n \xrightarrow{w} x$ if $\langle x_n, u \rangle \rightarrow \langle x, u \rangle$ and $x_n \xrightarrow{s} x$ if $\|x_n - x\| \rightarrow 0$) .

Definition (1.26) : [13]

Let T be an operator on a Hilbert space H , T is called essentially normal if $TT^* - T^*T$ is compact . It is well-known that every normal operator is an essentially normal.

Proposition(1.27) :

If $\gamma = 0$, then C_σ is an essentially normal composition operator .

Proof :

If $\gamma = 0$, then C_σ is a normal composition operator by (1.22) , by (1.26) C_σ is an essentially normal composition operator.

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حول الطبيعة الجوهرية للمؤثر التركيبي C_σ

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المستخلص

درسنا في هذا البحث المؤثر التركيبي المحتث من الدالة σ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المحتث من الدالة σ . بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير الدالة التحليلية σ .

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك برهنا بعض النتائج التي أعطيت بدون برهان.