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On Some Results for M-band Sub Filter Bank

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Abstract: In the present paper, we study aliasing signal, filter banks-band and sub M-band. Weobtain some results, like, the relationship between input and output signal, subsampling, input signal, output signal.

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Keywords: aliasing signal, filter banks, M-band.

1. Introduction:A filter bank is a signal processing device that produces M signals from a single signal by means of filtering by M simultaneous filters. The analysis filter bank splits the input signal $x(n)$ into a number of subband signals$x_k(n)$ **,** At the analysis stage, the input signal $x(n)$ is passed through a bank of M analysis filters$H_i(z)$, At the synthesis stage, the subbands are combined by a set of upsamplers and M synthesis filters $F_i(z)$ to form the reconstructed signal$\hat{x}(n)$[$2-7$].

Definition(1.1) [**1 − 5**]**:**A filter bank is called the perfect reconstruction filter bank if the reconstructed signal$\hat{x}(n)$ is a delayed or possibly scaled version of the original signal $x(n)$, i.e., $\hat{x}(n) = cx(n-d)$, $d \in Z , c \neq 0$.

Definition(1.2) [4]:The z-transform of a discrete-time signal x(n) is defined as:

$X(z) = \sum_{n=-\infty}^{+\infty} x(n)z^{-n}$ (1)

or, writing explicitly a few of the terms:

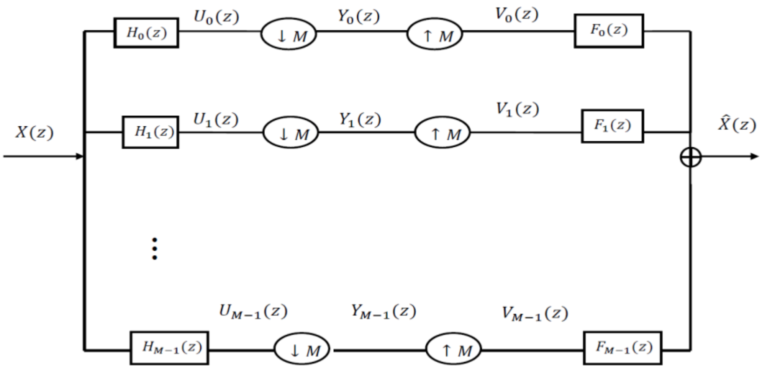
$X(z) = \cdots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots,$

where z is complex variable .

Remark:[6]The z-transform of $h(n)$ is called the transfer function of the filter and is defined by: $H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$ (Transfer function) (2)

1.1 Basic Filter Bank

Filter banks appear in two basic parts ,the first one is called analysis filter bank which can divides the signal into M filtered and downsampling input signal. Such filter bank is depicted in Fig (1)[6],and the second part is called synthesis filter bank generates a single signal from M upsampled and interpolated signals Fig.(1) shows such a synthesis filter bank[7].



Fig(1) Synthesis and analysis

The main idea of this structure is described as follow : The broadband signal $X(z)$ is split into M uniform sub-signals by analysis filter banks $H_0(z)$, $H_1(z)$, … , $H_{M-1}(z)$, (M is the number of subchannels) .Since bandwidth of sub-signals is narrower than the bandwidth of $X(z)$, the sample rate of sub-signals can be lowered by a factor M[7].

$$Y_i(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_i\left(W_M^k\, z\right)X(W_M^k\, z), \qquad (3)$$

where $i \in \{0,1,\dots,M-1\}$, $W_M = e^{-\frac{2\pi}{M}j}$

$$\therefore Y_i(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_i\left(e^{-\frac{2k\pi}{M}j}\, z\right)X\left(e^{-\frac{2k\pi}{M}j}\right), \qquad (4)$$

where $i = \{0,1 \dots., M-1\}$.

The reconstructed signal $\hat{X}(z)$ is obtained as :

$$\hat{X}(z) = \sum_{i=0}^{M-1} Y_i(z)F_i(z). \qquad (5)$$

Substituting Equation (4) into (5), we can have

$$\hat{X}(z) = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} H_i\left(e^{-\frac{2k\pi}{M}j}\, z\right) X(e^{-\frac{2k\pi}{M}j}\, z)F_i(z)$$

$$= \frac{1}{M}(\textstyle\sum_{i=0}^{M-1}(H_i(\,z)X(\,z)F_i(z) + H_i\left(e^{-\frac{2\pi}{M}j}\, z\right)X\left(e^{-\frac{2\pi}{M}j}\, z\right)F_i(z)$$

$$\cdots + H_i\left(e^{-\frac{2(M-1)\pi}{M}j}\, z\right)X(e^{-\frac{2(M-1)\pi}{M}j}\, z)F_i(z)). \qquad (6)$$

In the M-channel filter bank shown in Fig.(1), the reconstructedsignal is given by [4]

$$\hat{X}(z) = \frac{1}{M}\sum_{i=0}^{M-1} H_i(\,z)X(\,z)F_i(z)$$

$$+ \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=1}^{M-1} H_i\left(e^{-\frac{2k\pi}{M}j}\, z\right)X(e^{-\frac{2k\pi}{M}j}\, z)F_i(z),$$

where $T_k(\mathrm{z}) = \frac{1}{M}\sum_{k=1}^{M-1} X(e^{-\frac{2k\pi}{M}j}\, z)\sum_{i=0}^{M-1} H_i\left(e^{-\frac{2k\pi}{M}j}\, z\right)F_i(z)\ .$ (7)

In order to eliminate the aliasing error and guarantee the passband flat, the PR Subband filter banks should meet the condition as follows $[3-8]$:

$$T_k(\mathrm{z}) = 0 \quad where\ k = 1,2,3,\ldots,M-1\ , \qquad (7a)$$

and$H_0(\,z)F_0(z) + \cdots + H_{M-1}(\,z)F_{M-1}(z) = z^{-k_d}\ ,\ k_d \in N\ .$ $(7b)$

1.2 Modulation Matrices

The input-output relations of M-channel filter bank may also be written in matrix form. For this, we introduce the vector $X(z)$[8]

$X(z) = \begin{bmatrix} X(z) \\ X(zW_M\,) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$, where M is the number of sub channels.

And $H_m(z)$ bandpass analysis filters [6]

$$H(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M\,) & H_1(zW_M\,) & \cdots & H_{M-1}(zW_M\,) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix},$$

$F(z) = [F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)]\ .$

$X(z)$, $H(z)$ and $F(z)$ would be used in next.

2. Main Results:

Theorem (1): The relation between input and output signal can be: If $x(n) = e^{wnj}$is input signal, and $\hat{x}(n)$ is output signal defined in a way shown in Fig.(1), then the output signal will be defined as follows : $\hat{x}(n) = c\, z^{-d}\, x(n)$.

Proof: Since $x(n) = e^{wnj}$ (8) , $x(n)$is original signal ,and$\hat{x}(n)$ is delayed of the original signal $x(n)$.

Then by using Definition (1.1) and by Equation (8), we have $\hat{x}(n) = c\, e^{w(n-d)j}$

$\therefore \hat{x}(n) = c\, z^{-d}\, x(n)$. ■

Corollary (1): The original signal defined as following $x(n) = z\hat{x}(n)$, where $q \in R\, , d \in Z$.

Theorem (2): If $X(z)$ is z-transform of $x(n)$ and $\hat{X}(z)$ is z-transform of $\hat{x}(n)$ then:

1) The output signal $\hat{X}(z)$ as define $\hat{X}(z) = c\, z^{-d}\, X(z)$.

2) The input signal $X(z)$ as define$X(z) = q\, z^{d} \hat{X}(z)$.

Proof:

1) By Theorem (1), then$\hat{x}(n) = c\, z^{-d}\, x(n)$

$$\sum_{n=-\infty}^{\infty} \hat{x}(n) = cz^{-d} \sum_{n=-\infty}^{\infty} x(n)$$

$\sum_{n=-\infty}^{\infty} \hat{x}(n)\, z^{-n} = cz^{-d} \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \implies \hat{X}(z) = cz^{-d} X(z\,)$.

2) By proof (1)

$$\hat{X}(z) = cz^{-d} X(z\,) \quad \implies \quad X(z) = qz^{d} \hat{X}(z)\,. ■$$

Remark (1) If $x(n)$ is input signal, $\hat{x}(n)$ is output signal, then the aliasing in $x(n)$, $\hat{x}(n)$is cz^{-d}, such that z is complex number .

Proof: Let aliasing is $\rho$and$\hat{x}(n) = \rho x(n)$.(9)

By Theorem (1), we have $\hat{x}(n) = cz^{-d} x(n)$. (10)

Form Equation (9) in Equation (10),we obtain

$$\rho x(n) = cz^{-d} x(n) \quad \Rightarrow \rho = cz^{-d}$$

$\therefore$ The aliasing is cz^{-d} .

Theorem (3): If H (z) is analysis filter and $X(z)$ is input signal,then the subsampling by M is $Y(z) = \frac{1}{M} H^{T}(z) X(z)$ if and only if $Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} H_i\big(zW_M^k\big) X(zW_M^k)$.

Proof: $(1 \Longrightarrow 2)$Let $Y(z) = \frac{1}{M} H^T(z) X(z)$

$$= \frac{1}{M} \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M^1) & H_1(zW_M^1) & \cdots & H_{M-1}(zW_M^1) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix} \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M} \left[\sum_{k=0}^{M-1} H_0(zW_M^k) X(zW_M^k) + \sum_{k=0}^{M-1} H_1(zW_M^k) X(zW_M^k) \right.$$

$$\left. + \cdots + \sum_{k=0}^{M-1} H_0(zW_M^k) X(zW_M^k) \right]$$

$\therefore Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} H_i(zW_M^k) X(zW_M^k)$.

$(2 \Longrightarrow 1)$Let $Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} H_i(zW_M^k) X(zW_M^k)$.

$$= \frac{1}{M} (H_0(z)X(z) + H_0(zW_M)X(zW_M) + \cdots + H_0(zW_M^{M-1})$$

$$X(zW_M^{M-1}) + \cdots + H_{M-1}(z)X(z) + H_{M-1}(zW_M)X(zW_M)$$

$$+ \cdots + H_{M-1}(zW_M^{M-1})X(zW_M^{M-1}))$$

$$= \frac{1}{M} \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M) & H_1(zW_M) & & H_{M-1}(zW_M) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}^T \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M} H^T(z) X(z). \blacksquare$$

Theorem (4): If $H(z)$ is analysis filter, $F(z)$ is synthesis filter and $X(z)$ is input signal,then the output signal is $\hat{X}(z) = \frac{1}{M} F(z) H^T(z) X(z)$ if and only if

$$\hat{X}(z) = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} F_i(z) H_i(zW_M^k) X(zW_M^k)$$

Proof: $(1 \Longrightarrow 2)$.Let $\hat{X}(z) = \frac{1}{M} F(z) H^T(z) X(z)$. Then

$$\hat{X}(z) = \frac{1}{M} [F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)]$$

$$\cdot \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M) & H_1(zW_M) & \cdots & H_{M-1}(zW_M) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix} \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \left[\sum_{i=0}^{M-1} F_i(z)H_i(z) \quad \sum_{i=0}^{M-1} F_i(z)H_i(zW_M) \quad \cdots \right.$$

$$\left. \sum_{i=0}^{M-1} F_i(z)H_i(zW_M^{M-1}) \right] \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M} \sum_{i=0}^{M-1} F_i(z)H_i(z)X(z)$$

$$+ \left[\sum_{i=0}^{M-1} F_i(z)H_i(zW_M)X(zW_M) + \cdots + \sum_{i=0}^{M-1} F_i(z)H_i(zW_M^{M-1})X(zW_M^{M-1}) \right]$$

$$= \frac{1}{M} \sum_{i=0}^{M-1} \sum_{k=0}^{M-1} F_i(z)H_i(zW_M^k)\, X(zW_M^k) \qquad (11)$$

$(2 \Longrightarrow 1)$Let $\hat{X}(z) = \frac{1}{M} \sum_{i=0}^{M-1} \sum_{k=0}^{M-1} F_i(z)H_i(zW_M^k)\, X(zW_M^k)$

$$= \frac{1}{M} \left(\sum_{k=0}^{M-1} F_0(z)H_0(zW_M^k)X(zW_M^k) + \sum_{k=0}^{M-1} F_1(z)H_1(zW_M^k) \right.$$

$$\left. X(zW_M^k) + \cdots + \sum_{k=0}^{M-1} F_0(z)H_0(zW_M^k)X(zW_M^k) \right)$$

$$= \frac{1}{M} [F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)] \begin{bmatrix} \sum_{k=0}^{M-1} H_0(zW_M^k)X(zW_M^k) \\ \sum_{k=0}^{M-1} H_1(zW_M^k)X(zW_M^k) \\ \vdots \\ \sum_{k=0}^{M-1} H_{M-1}(zW_M^k)X(zW_M^k) \end{bmatrix}$$

$$= \frac{1}{M} [F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)]$$

$$\begin{bmatrix} H_0(z)X(z) + H_0(zW_M)X(zW_M) + \cdots + H_0(zW_M^{M-1})X(zW_M^{M-1}) \\ H_1(z)X(z) + H_1(zW_M)X(zW_M) + \cdots + H_1(zW_M^{M-1})X(zW_M^{M-1}) \\ \vdots \\ H_{M-1}(z)X(z) + H_{M-1}(zW_M)X(zW_M) + \cdots + H_{M-1}(zW_M^{M-1})X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M} [F_0(z) \quad F_1(z) \quad \cdots \quad F_{M-1}(z)]$$

$$\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW_M) & H_1(zW_M) & \cdots & H_{M-1}(zW_M) \\ \vdots & & & \\ H_0(zW_M^{M-1}) & H_1(zW_M^{M-1}) & \cdots & H_{M-1}(zW_M^{M-1}) \end{bmatrix}^T \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}$$

$$= \frac{1}{M} F(z)H^T(z)X(z). \blacksquare$$

Theorem (5): If $X(z)$ is input signal and reconstructed without distortions, $H(z)$ is analysis filter and $F(z)$ is synthesis filter, then$\frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z) = cz^{-d}$, such that z is complex number, $\neq 0$.

Proof: Let $X(z)$ is reconstructed without distortions

By Theorem (2) we get$\hat{X}(z) = c\, z^{-d}\, X(z)$, (12)

fromEquation (6) inEquation (12) we get

$$\frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z) + \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=1}^{M-1} F_i(z)H_i\left(zW_M^k\right) X\left(zW_M^k\right) = c\, z^{-d}\, X(z)$$

$$\therefore\ \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z) = c\, z^{-d}. \blacksquare$$

Theorem (6): If $H(z)$ is analysis filter, $F(z)$ is synthesis filter, z is complex number and $\neq 0$, then $c\, z^{-d} = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)$ if and only if

$c\, z^{-d}X(z) = F(z)Y(z) - \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\left(zW_M^k\right) X(z\, W_M^k)$.

Proof: Let $z^{-d} = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)$,

$$c\, z^{-d}X(z) = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z)$$

$$= (\frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z) + \frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1} F_i(z)H_i\left(zW_M^k\right)X\left(zW_M^k\right))$$

$$-\frac{1}{M}\sum_{i=0}^{M-1}\sum_{k=0}^{M-1} F_i(z)H_i\left(zW_M^k\right)X\left(zW_M^k\right).$$

By Theorem (3) we get

$c\, z^{-d}X(z) = F(z)Y(z) - \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\left(zW_M^k\right) X(z\, W_M^k)$.

Now, let $c\, z^{-d}X(z) = F(z)Y(z) - \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\left(zW_M^k\right) X(z\, W_M^k)$and

let$\lambda = F(z)Y(z)$

By Theorem (3) we get

$\lambda = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)X(z) + \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\left(zW_M^k\right) X(z\, W_M^k)$.

Now

$$c\, z^{-d}X(z) = \lambda - \frac{1}{M}\sum_{k=0}^{M-1}\sum_{i=0}^{M-1} F_i(z)H_i\left(zW_M^k\right) X(z\, W_M^k)$$

$\therefore\, c\, z^{-d} = \frac{1}{M}\sum_{i=0}^{M-1} F_i(z)H_i(z)$.■

Conclusion:

In this paper we find a relation between input signal and output signal and from that we can find the aliasing part in the system.

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