On a New Class of Meromorphic Univalent Function Associated with Dziok_Srivastava Operator

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Abstract

In this paper, we introduce and study a new class of meromorphic Univalent functions defined by Dziok_Srivastava operator for this class. We obtain coefficient inequality, convex set, closure and Hadamard product (or convolution).Further we obtain a(n,δ)-neighborhood of the function $f \in \vartheta$, and the integral transform.

Keywords

Meromorphic Univalent function, Hadamard product (or convolution), Neighborhood, Integral transform, Convex set.

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1. Introduction

Let ϑ denote the class of functions of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \quad , \quad (a_k \geq 0, k \in \mathbb{N} = \{1,2,\dots\}) . \qquad (1.1)$$

Which are analytic Meromorphic Univalent in the punctured unit disk $U^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = U\backslash 0$. Let $\Im(\beta,\alpha,\mu,\lambda)$ be a subclass of ϑ of function of the form

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \quad , \quad (a_k \geq 0, k \in \mathbb{N} = \{1,2,\dots\}). \qquad (1.2)$$

The Hadamard product (or convolution) of two power series

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \quad ,$$

$$g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k \quad , (1.3)$$

in ϑ is defined (as usual) by

$$(f*g)(z)=f(z)*g(z)$$
$$=z^{-1}+\sum_{k=1}^{\infty}a_kb_kz^k\ . \quad (1.4)$$

Corresponding to the function $h(a_1,\dots,a_q;b_1,\dots,b_s;z)=z_q^{-1}F_s(a_1,\dots,a_q;b_1,\dots,b_s;z)$.

The operator

$$\left(H_s^q[a_1]f(z)\right)$$
$$=\left(H_s^q(a_1,\dots,a_q;b_1,\dots,b_s)\right)f(z)$$
$$=h(a_1,\dots,a_q;b_1,\dots,b_s;z)*f(z)$$
$$=z^{-1}+\sum_{k=n+1}^{\infty}\frac{(a_1)_{k-1}\dots(a_q)_{k-1}a_k}{(b_1)_{k-1}\dots(b_s)_{k-1}(k-1)!}z^k$$
$$=z^{-1}+\sum_{k=n+1}^{\infty}\mathrm{h}(k)a_kz^k\ . \quad (1.5)$$

Where $*$ stands for convolution of two power series, $f\epsilon s$ and

$$\mathrm{h}(k)=\frac{(a_1)_{k-1}\dots(a_q)_{k-1}a_k}{(b_1)_{k-1}\dots(b_s)_{k-1}(k-1)!}. \quad (1.6)$$

Here $qF_s(Z)$ is the generalized hypergeometric function for $a_j\in\mathbb{C}\ (j=1,2,3,\dots,q)$ and $b_j\in\mathbb{C}\ (j=1,2,3,\dots,s)$ Such that $b_j\neq 0,-1,-2,\dots(j=1,2,3,\dots,s)$ defined by

$$qF_s(Z)=F_s(a_1,\dots,a_q;b_1,\dots,b_s;z)$$
$$=\sum_{k=n+1}^{\infty}\frac{(a_1)_k\dots(a_q)_k}{(b_1)_k\dots(b_s)_kk!}z^k,\ (q\le s+1,q,s\in\mathbb{N}_0,z\in U)\ , \quad (1.7)$$

where $(\lambda)_n$is the Pochhammer symbol defined, in terms of gamma function by

$$(\lambda)_n=\frac{\Gamma(\lambda+\mathrm{n})}{\Gamma(\lambda)}=\begin{cases}1 & (n=0)\\ \lambda(\lambda+1),\dots,(\lambda+\mathrm{n}-1) & (n\in\mathbb{N})\end{cases}.$$

The series $qF_s(Z)$ in (1.7) converges absolutely for$|z|<\infty$, if $q<s+1$ and for$|z|=1$, if $q=s+1$.The linear operator defined in (1.5) is the Dziok_Srivastav operator see [8].Which contains the well define operators like Seoudy and Aouf [10], and see Dziok, Murugusundaramoorthy and Sokot[6], the saitoh generalized linear operator, the Bernardi_Libera Livingston operator and many others.

Definition(1.1) Let $f\in\vartheta$ be given (1.2).The class $\Im(\beta,\alpha,\mu,\lambda)$is defined by the subclass of a consisting of functions of the form (1.2) and satisfying the analytic criterion:

$$\Im(\beta,\alpha,\mu,\lambda)=\left|\frac{z\left(H_s^q[\alpha_1]f(z)\right)''+2\left(H_s^q[\alpha_1]f(z)\right)'}{\mu z\left(H_s^q[\alpha_1]f(z)\right)''+\alpha\left(H_s^q[\alpha_1]f(z)\right)'}\right|<\beta, \quad (1.8)$$

for $0<\beta\le 1$,$0<\alpha<\mu<1$, $a_j\in\mathbb{C}(j=1,2,3,\dots,q)$ and $b_j\in\mathbb{C}(j=1,2,3,\dots,s)$, such that $b_j\neq 0,-1,-2,\dots(j=1,2,3,\dots,s)$, $(q\le s+1.q,s\in\mathbb{N},z\in U)$.

The present paper aims at providing a systematic investigation of various interesting properties and characteristics of function belonging to the new class $\Im(\beta,\alpha,\mu,\lambda)$. The properties such as the neighborhood, convex set, Hadamard product and integral operator defined on the new class $\Im(\beta,\alpha,\mu,\lambda)$are also discussed.

Atshan [3], Cho et al. [5], Atshan and Kulkarni [4] and Aouf [2,10] are studied the meromorphic univalent function for different classes.

2. Coefficients Inequalities

First, in the following theorem, we obtain necessary and sufficient condition for a function f to be in the class $\Im\,(\beta,\alpha,\mu,\lambda)$.

Theorem(2.1) Let $f \in \vartheta$ be given by (1.2).Then $f \in \Im\,(\beta,\alpha,\mu,\lambda)$ if and only if

$$\sum_{k=n+1}^{\infty} h(k)k[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)]\,a_k \leq \beta(2\mu-\alpha)\,, \qquad (2.1)$$

for $\quad 0<\beta\leq 1\,, 0<\alpha<\mu<1.$

The result is sharp for the function

$$f(z)=z^{-1}+\frac{\beta(2\mu-\alpha)}{h(k)k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]}z^k,$$

$k\geq 1.$

Proof:- Assume that the inequality (2.1) holds true and let $|z|=1$, then from (1.8), we have

$$\left|z\left(H_s^q[\alpha_1]f(z)\right)''+2\left(H_s^q[\alpha_1]f(z)\right)'\right|-\beta\left|\mu z\left(H_s^q[\alpha_1]f(z)\right)''+\alpha\left(H_s^q[\alpha_1]f(z)\right)'\right|$$

$$=\left|\left(2z^{-2}+\sum_{k=n+1}^{\infty}h(k)\,a_k k(k-1)z^{k-1}\right)+2\left(-z^{-2}+\sum_{k=n+1}^{\infty}h(k)\,a_k kz^{k-1}\right)\right|-\beta\left|\mu\left(2z^{-2}+\sum_{k=n+1}^{\infty}h(k)\,a_k k(k-1)z^{k-1}\right)+\alpha\left(-z^{-2}+\sum_{k=n+1}^{\infty}h(k)\,a_k kz^{k-1}\right)\right|$$

$$=\left|\sum_{k=n+1}^{\infty}h(k)\,a_k k(k+1)z^{k-1}\right|-\beta\left|(z^{-2}(2\mu-\alpha)+\sum_{k=n+1}^{\infty}h(k)\,a_k kz^{k-1}(\mu(k-1)+\alpha)\right| \qquad (2.2)$$

$$\leq\sum_{k=n+1}^{\infty}h(k)\,k(k+1-\beta\mu k+\beta\mu-\beta\beta\alpha)a_k-\beta(2\mu-\alpha)\leq 0.$$

Therefore,

$$\sum_{k=n+1}^{\infty}h(k)\,k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]a_k \leq \beta(2\mu-\alpha)$$

Hence, by maximum modulus principle, $f \in \Im\,(\beta,\alpha,\mu,\lambda)$.

Conversely, suppose that $f \in \Im\,(\beta,\alpha,\mu,\lambda)$, then from (2.2), we have

$$\left|\frac{z\left(H_s^q[\alpha_1]f(z)\right)''+2\left(H_s^q[\alpha_1]f(z)\right)'}{\mu z\left(H_s^q[\alpha_1]f(z)\right)''+\alpha\left(H_s^q[\alpha_1]f(z)\right)'}\right|$$

$$=\left|\frac{\sum_{k=n+1}^{\infty}h(k)a_k k(k+1)z^{k-1}}{z^{-1}(2\mu-\alpha)+\sum_{k=n+1}^{\infty}h(k)a_k kz^{k-1}(\mu(k-1)+\alpha)}\right|$$

$<\beta$.

Since $|Re(z)|\leq|z|$ for all z , we have

$$Re\left\{\frac{\sum_{k=n+1}^{\infty}h(k)a_k k(k+1)z^{k-1}}{z^{-1}(2\mu-\alpha)+\sum_{k=n+1}^{\infty}h(k)a_k kz^{k-1}(\mu(k-1)+\alpha)}\right\}$$

$<\beta$.

We can choose the value of z on the real axis, so that $z\left(H_s^q[\alpha_1]f(z)\right)''$ *and* $\left(H_s^q[\alpha_1]f(z)\right)'$ are real, upon clearly the denominator of (1.10) and letting $z\to 1^-$, through real values . We get the inequality (2.1). Sharpness of the result follows be setting

$$f(z)=z^{-1}+\frac{\beta(2\mu-\alpha)}{h(k)k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]}z^k,$$

$$k\geq 1. \qquad (2.3)$$

The proof is complete.■

Corollary (2.4) Let $f \in \mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$.Then

$$a_k \leq \frac{\beta(2\mu-\alpha)}{h(k)k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]}, \qquad (2.4)$$

where $0<\beta\leq 1, 0<\alpha<\mu<1$.

3.Closure on $\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$

Let the function f be defined by

$$f(z)=z^{-1}+\sum_{k=1}^{\infty} a_k z^k.$$

Now, we shall prove the following result for the closure of such a function in the class $\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$.

Theorem (3.1) Let the functions f_r defined by $f_r(z)=z^{-1}+\sum_{k=n+1}^{\infty} a_{k,r}z^k$,

$(a_{k,r}\geq 0, k\in\mathbb{N}=\{1,2,\dots\}, \mathrm{r}=1,2,3,\dots,\mathrm{l})$ be in the class $\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$ for every r=1,2,3,…,l. Then the function h defined by

$$h(z)=z^{-1}+\sum_{k=n+1}^{\infty} e_k z^k .(\,e_k\geq 0, k\in\mathbb{N})$$

Also belongs to the class $\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$, where

$$e_k=\frac{1}{l}\sum_{r=1}^{l} a_{k,r}\ ,(\,k=1,2,3,\dots).$$

Proof:-Since $f_k\in\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$ it follows from Theorem (2.1) that

$$\sum_{k=n+1}^{\infty} h(k)\,k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]a_{r,k}\leq\beta(2\mu-\alpha),$$

for every r =1,2,…l.

Hence,

$$\sum_{k=n+1}^{\infty} h(k)\,k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]e_k$$

$$=\sum_{k=n+1}^{\infty} h(k)\,k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]\left(\frac{1}{l}\sum_{r=1}^{l} a_{k,r}\right)$$

$$=\frac{1}{l}\sum_{r=1}^{l}\ \left(\sum_{k=n+1}^{\infty} h(k)\,k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]a_{k,r}\right)\leq\beta(2\mu-\alpha).$$

Hence, by Theorem (2.1), it follows that $h\in\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$.

The proof is complete.■

4. Convex Set

Now, we state a theorem of convex set of the functions $(1-\gamma)f(z)+\gamma g(z)$ in the class $\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$.

Theorem (4.1) The class $\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$ is convex set.

Proof:- let f and g be the arbitrary of $\mathfrak{I}\,(\beta,\alpha,\mu,\lambda)$.Then for every γ

$(0<\gamma<1)$, we show that

$$(1-\gamma)f(z)+\gamma g(z)\in\mathfrak{I}\,(\beta,\alpha,\mu,\lambda).$$

Thus we have

$$(1-\gamma)f(z)+\gamma g(z)=z^{-1}+\sum_{k=n+1}^{\infty}[(1-\gamma)a_k+\gamma b_k],$$

and

$$\frac{\sum_{k=n+1}^{\infty} h(k)k[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)]a_k}{\beta(2\mu-\alpha)}[(1-\gamma)a_k+\gamma b_k],$$

$$=(1-\gamma)\frac{\sum_{k=n+1}^{\infty} h(k)k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]}{\beta(2\mu-\alpha)}a_k+\gamma\frac{\sum_{k=n+1}^{\infty} h(k)k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]}{\beta(2\mu-\alpha)}b_k\leq 1\,.$$

The proof is complete.■

5. Hadamard Product

We consider the Hadamard product (or convolution) of two power series

$$f(z)=z^{-1}+\sum_{k=1}^{\infty} a_k z^k\ \ ,\ \ g(z)=z^{-1}+\sum_{k=1}^{\infty} b_k z^k\,,$$

in ϑ is defined (as usual) by

$$(f*g)(z)=f(z)*g(z)=z^{-1}+\sum_{k=1}^{\infty}a_kb_kz^k .$$

Theorem (5.1) Let $f,g \in \Im(\beta,\alpha,\mu,\lambda)$.Then $f*g \in \Im(\beta,\alpha,\mu,\lambda)$ for

$$f(z)=z^{-1}+\sum_{k=n+1}^{\infty}a_kz^k \quad , \quad g(z)=z^{-1}+\sum_{k=n+1}^{\infty}b_kz^k$$

and

$$(f*g)(z)=z^{-1}+\sum_{k=n+1}^{\infty}a_kb_kz^k,$$

where

$$\delta=\frac{\beta^2(2\mu-\alpha)(k+1)}{\beta^2(2\mu-\alpha)(\mu(k-1)+\alpha)+h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]^2}.$$

Proof:- Since f and g are in the class $\Im(\beta,\alpha,\mu,\lambda)$, then

$$\frac{\sum_{k=n+1}^{\infty}h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]a_k}{\beta(2\mu-\alpha)}\le 1, \quad (5.1)$$

and

$$\frac{\sum_{k=n+1}^{\infty}h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]b_k}{\beta(2\mu-\alpha)}\le 1 . \quad (5.2)$$

We have to find the largest δ such that

$$\frac{\sum_{k=n+1}^{\infty}h(k)k[k(1-\delta\mu)+(1+\delta(\mu-\alpha))]}{\delta(2\mu-\alpha)}a_kb_k\le 1. \quad (5.3)$$

By Cauchy-Schwarz inequality, we get

$$\frac{\sum_{k=n+1}^{\infty}h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]}{\beta(2\mu-\alpha)}\sqrt{a_kb_k}\le 1. \quad (5.4)$$

We want only to show that

$$\frac{\sum_{k=n+1}^{\infty}h(k)k[k(1-\delta\mu)+(1+\delta(\mu-\alpha))]}{\delta(2\mu-\alpha)}a_kb_k\le \frac{\sum_{k=n+1}^{\infty}h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]}{\beta(2\mu-\alpha)}\sqrt{a_kb_k} .$$

This equivalently to

$$\sqrt{a_kb_k}\le\frac{\delta([(k(1-\beta\mu)+(1+\beta(\mu-\alpha))])}{\beta([k(1-\delta\mu)+(1+\delta(\mu-\alpha))])} . \quad (5.5)$$

From (5.4), we get

$$\sqrt{a_kb_k}\le\frac{\beta(2\mu-\alpha)}{h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]}.$$

Thus it is enough to show that

$$\frac{\beta(2\mu-\alpha)}{h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]}\le\frac{(\delta[(k(1-\beta\mu)+(1+\beta(\mu-\alpha))])}{(\beta[k(1-\delta\mu)+(1+\delta(\mu-\alpha))])},$$

which simplifies to

$$\delta=\frac{\beta^2(2\mu-\alpha)(k+1)}{\beta^2(2\mu-\alpha)(\mu(k-1)+\alpha)+h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]^2}.$$

The proof is complete.■

6. Bounds Theorem

We consider the function $f_j(j=1,2)$ defined by

$$f_j(z)=z^{-1}+\sum_{k=n+1}^{\infty}a_{k,j}z^k ,(a_{k,j}\ge 0, j=1,2)$$

be in the class $\Im(\beta,\alpha,\mu,\lambda)$. Such that the function h defined by

$$h(z)=z^{-1}+\sum_{k=2}^{\infty}({a_{k,1}}^2+{a_{k,2}}^2)z^k$$

,

in the class$\Im(\beta,\alpha,\mu,\lambda)$ for η.

Theorem (6.1)

Let the function $f_j (j = 1,2)$ defined by

$$f_j(z) = z^{-1} + \sum_{k=n+1}^{\infty} a_{k,j} z^k , (a_{k,j} \geq 0, j = 1,2)$$

be in the class $\Im (\beta, \alpha, \mu, \lambda)$. Then the function h defined by

$$h(z) = z^{-1} + \sum_{k=2}^{\infty} ({a_{k,1}}^2 + {a_{k,2}}^2) z^k \quad , \quad (6.1)$$

belongs to the $\Im (\beta, \alpha, \mu, \lambda)$ where

$$\eta = \frac{\beta^2(2\mu-\alpha)(k+1)}{\beta^2(2\mu-\alpha)(\mu(k-1)+\alpha)+h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]^2}.$$

Proof:- Note that

$$\sum_{k=n+1}^{\infty} \left[\frac{h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]}{\beta(2\mu-\alpha)} \right]^2 {a_{k,j}}^2$$

$$\leq \left[\sum_{k=n+1}^{\infty} \frac{h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]}{\beta(2\mu-\alpha)} a_{k,j} \right]^2 \leq 1 \;. (j = 1,2) \quad (6.2)$$

For $f_j \in \Im (\beta, \alpha, \mu, \lambda)$, we have

$$\sum_{k=n+1}^{\infty} \frac{1}{2} \left(\frac{h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]}{\beta(2\mu-\alpha)} \right)^2 ({a_{k,1}}^2 + {a_{k,2}}^2) \leq 1. \quad (6.3)$$

In order to obtain our result we have to find the largest η such that

$$\frac{[k(1-\eta\mu)+(1+\eta(\mu-\alpha))]}{\eta} \leq \frac{h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]^2}{\beta^2(2\mu-\alpha)}. \quad (k \geq 1)$$

So that

$$\eta = \frac{\beta^2(2\mu-\alpha)(k+1)}{\beta^2(2\mu-\alpha)(\mu(k-1)+\alpha)+h(k)k[k(1-\beta\mu)+(1+\beta(\mu-\alpha))]^2}.$$

The proof is complete.■

7. Neighborhoods

The concept of neighborhood of analytic functions was first introduced by Goodman [7] and Ruscheweyh [9] investigated this concept for the elements of several famous subclasses of analytic functions and Altintas and Owa [1] considered for a certain family of analytic functions with negative coefficients, also Liu and Srivastava [8] and Atshan [3] extended this concept for a certain subclass of meromorphically univalent and multivalent functions.

Now, we define the (k, δ)-neighborhood of a function $f \in \vartheta$ by

$$N_{k,\delta}(f) = \{g \in \vartheta : g(z) = z^{-1} + \sum_{k=n+1}^{\infty} b_k z^k \;\; and \;\; \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta , 0 \leq \delta < 1\}. \quad (7.1)$$

For the identity function $e(z) = z$, we have

$$N_{k,\delta}(e) = \left\{ g \in \vartheta : g(z) = z^{-1} + \sum_{k=n+1}^{\infty} b_k z^k \;\; and \;\; \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}.$$

Definition (7.1) A function $f \in \vartheta$ is said to be in the class $\Im (\beta, \alpha, \mu, \lambda)$ if there exists a function $g \in \Im (\beta, \alpha, \mu, \lambda)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \eta \quad , (z \in U, 0 \leq \eta < 1).$$

Theorem (7.1) If $g \in \Im (\beta, \alpha, \mu, \lambda)$ and

$$\eta = 1 - \frac{\delta(2-\beta\alpha)}{(2-\beta\alpha)-\beta(2\mu-\alpha)} . \quad (7.2)$$

Then $N_{k,\delta}(g) \subset \Im\,(\beta,\alpha,\mu,\lambda)$.

Proof:- Let $f \in N_{k,\delta}(g)$. We want to find from (7.1) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \le \delta\,,$$

which readily implies the following coefficient inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \le \delta\,, (k \in \mathbb{N}). \quad (7.3)$$

Next, since $g \in \Im\,(\beta,\alpha,\mu,\lambda)$, we have from Theorem (2.1)

$$\sum_{k=n+1}^{\infty} b_k \le \frac{\beta(2\mu-\alpha)}{(2-\beta\alpha)}. \quad (7.4)$$

So that

$$\left|\frac{f(z)}{g(z)} - 1\right| \le \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty} b_k} \le \frac{\delta(2-\beta\alpha)}{(2-\beta\alpha)-\beta(2\mu-\alpha)} = 1-\eta\,.$$

Thus by Definition (7.1), $f \in \Im\,(\beta,\alpha,\mu,\lambda)$ for η given by (7.2).

The proof is complete.■

8. Integral Transforms

Next, we consider integral transforms of functions in the class $\Im\,(\beta,\alpha,\mu,\lambda)$, some of these integral transforms was studied by Atshan on the other class in [3].

Theorem (8.1) Let the function f given by (1.2) be in the class $\Im\,(\beta,\alpha,\mu,\lambda)$. Then the integral operator

$$F(z) = c\int_0^1 u^c f(uz)\,du\,,\ (0 < u \le 1\,, 0 < c < \infty), \quad (8.1)$$

is in the class $\Im(\beta,\alpha,\mu,\lambda)$.

Where

$$\gamma = \frac{\beta c(k+1)}{((c+k+1)(k(1-\beta\mu)+(1+\beta(\mu-\alpha)))+\beta c(\mu(k-1)+\alpha))}.$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{\beta(2\mu-\alpha)}{(1-\beta\mu)+(1+\beta(\mu-\alpha))} z\,.$$

Proof:- Let

$$f(z) = z^{-1} + \sum_{k=n+1}^{\infty} a_k z^k\,.$$

In the class $\Im\,(\beta,\alpha,\mu,\lambda)$.Then

$$F(z) = c\int_0^1 u^c f(uz)\,du$$

$$= c\int_0^1 \left(u^{c-1}z^{-1} + \sum_{n=1}^{\infty} a_k u^{k+c} z^k\right) du$$

$$= z^{-1} + \sum_{n=1}^{\infty} \frac{c}{c+k+1} a_k z^k\,. \quad (8.2)$$

It is sufficient to show that

$$\frac{\sum_{k=n+1}^{\infty} c\,h(k)k\left[k(1-\gamma\mu)+\left(1+\gamma(\mu-\alpha)\right)\right]a_k}{(c+k+1)\gamma(2\mu-\alpha)} \le 1\,. \quad (8.3)$$

Since $f \in \Im\,(\beta,\alpha,\mu,\lambda)$, we have

$$\frac{\sum_{k=n+1}^{\infty} h(k)k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]a_k}{\beta(2\mu-\alpha)} \le 1\,.$$

Note that (8.3) it satisfied if

$$\frac{c\,h(k)k\left[k(1-\gamma\mu)+\left(1+\gamma(\mu-\alpha)\right)\right]a_k}{(c+k+1)\gamma(2\mu-\alpha)} \le \frac{h(k)k\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right]a_k}{\beta(2\mu-\alpha)}.$$

Rewriting the inequality, we have

$$(c+k+1)\gamma\left[k(1-\beta\mu)+\left(1+\beta(\mu-\alpha)\right)\right] \le \beta c\left[k(1-\gamma\mu)+\left(1+\gamma(\mu-\alpha)\right)\right].$$

Solving for γ, we have

$$\gamma = \frac{\beta c(k+1)}{((c+k+1)(k(1-\beta\mu)+(1+\beta(\mu-\alpha)))+\beta c(\mu(k-1)+\alpha))} = F(k). \quad (8.4)$$

A simple computation will show that $F(n)$ is increasing $F(k) \geq F(1)$. Using this, the result follows.

The proof is complete.■

References

[1] O. Altintas and S. Owa, ***Neighborhoods of certain analytic functions with negative coefficients,*** Int. J. Math. Sci., 19, **(1996),** 797-800.

[2] M. K. Aouf, ***On a certain class of meromorphic univalent functions with positive coefficients,*** Rendiconti di Math. Serie V11, vol. 11, Roma, **(1981)**, 209_219.

[3] W. G. Atshan, ***Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative II*** , J. Surveys in Mathematics and its Applications , 3, **(2008),** 67-77.

[4] W. G. Atshan and S. R. Kulkarni, ***On a class of P_valent analytic function with negative coefficients defined by Dziok_Srivastava linear operator,*** International J. Math. Sci. & Engg. Appls. (IJMSEA],2(1), **(2007)**, 173-182.

[5] N. E. Cho, H. Lee and S. Owa, ***A class of meromorphic univalent functions with positive coefficients,*** Koebeg, Math. ,4, **(1987)**, 43_50.

[6] J. Dziok, G. Murugusundaramoorthy and J. Sokot', *On certain class of meromorphic functions with positive coefficients,* Acta. Mathematica scientia, 32 **(2012)**, B(4):1376_1390.

[7] A. W. Goodman, *Univalent functions and non-analytic curves*, Proc. Amer. Math. Soc., 8, **(1975),** 598-601.

[8] J. L. Liu and H. M. Srivastava, ***A linear operator and associated families of meromorphically multivalent functions,*** J. Math. Anal. Appl. 259, **(2001),** 566-581.

[9] S. Ruscheweyh, ***Neighborhoods of Univalent functions,*** Proc. Amer. Math. Soc., 81, **(1981)**, 521-527.

[10] T. M. Seoudy and M. K. Aouf**, *Some inequalities of meromorphic P_valent functions associated with the Liu-Srivastava operator,* (2013),** proceedings of IAM, V. 2, n.1, PP. (11_9).

دراسة على صنف جديد من الدوال الاحادية التكافؤ الميرومورفية والمعرفة بواسطة مؤثر ديزويك_سرفستيفه

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المستخلص

في هذه الورقة، قدمنا ودرسنا صنف جديد من الدوال احادية التكافؤ الميرومورفية والمعرفة بواسطة مؤثر ديزويك_سرفستيفه على هذا الصنف. وحصلنا على متراجحة المعامل، النقاط الشاذة، مبرهنة الانغلاق، شرط الالتفاف، وكذلك حصلنا على خاصية الجوار للدالة f من النمط (n,δ) وتمثيل التكامل.