ISSN 2347-1921

On Some Results of a New Fractional Calculus in the Unit Disk and its Applications

Hisham Mohammed Ali*, Waggas Galib Atshan** and Duaa Mohammed Hamid***
Department of Mathematics, College of Computer Science & Mathematics,
University of Al-Qadisiya, Diwaniya – Iraq.
*houshammah@gmail.com,**waggashnd@gmail.com,***duaa.m.kh@gmail.com .

Abstract:

In this paper, we study a new definition of fractional calculus (fractional derivative-(β,γ),fractional integral-(β,γ)),by depending of Srivastava-Owa fractional operators,and study some properties and provided of operators-(β,γ),and Applecations by using fractional complex transform for fractionaldifferential equations , Distortion inequalities and establishing the sufficient conditions for the existence and uniqueness of Cauchy problem in the unit disk.

2015Mathematics Subject Classification: 30C45

Keywords: fractional calculus(fractional derivative and fractional integral); Srivastava-Owa fractional operators, analytic function; fractional complex transform; fractional differential equations; fractional complex transform; Cauchy differential equation and Fox-Wright function .

1 INTRODUCTION

Definition(1.1)[1,2,3,4]: The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\alpha}} d\zeta ; \quad 0 \le \alpha < 1 ,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of$(z-\zeta)^{-\alpha}$ is removed by requiring log$(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition (1.2)[1,2,3,4]: The fractional integral of order α is defined, for a function ,

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} \, d\zeta ; \quad 0 \le \alpha < 1 ,$$

where the function $f(z)$is analytic in simply connected region of the complex z-plane $\mathbb{C}$containing the origin, and the multiplicity of$(z-\zeta)^{\alpha-1}$is removed by requiring log$(z-\zeta)$ to be real when$(z-\zeta)$ >0.

Remark(1.1)[1,2,3,4]: From Definitions 1.1 and 1.2 ,we have

$$D_z^{\alpha}\{z^{v}\} = \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} z^{v-\alpha} \; ; v > -1 , 0 \le \alpha < 1$$

and

$$I_z^{\alpha}\{z^{v}\} = \frac{\Gamma(v+1)}{\Gamma(v+1+\alpha)} z^{v+\alpha} \; ; v > -1 , \quad \alpha > 0$$

Remark(1.2): From above remark we have,

$$I_z^{\alpha} f(z) = D_z^{-\alpha} f(z). \tag{1}$$

2 Fractional Calculus-(β,γ)

Now, we define the a new fractional integral operator $I_z^{\alpha,\beta,\gamma} f(z)$ .Consider for natural number n∈N={1,2,...} and real β,γ then-fold integral of the form

$$I_z^{\alpha,\beta,\gamma} f(z) = \int_0^z {\zeta_1}^{\beta-\gamma-\frac{1}{2}} d\zeta_1 \int_0^{\zeta_1} {\zeta_2}^{\beta-\gamma-\frac{1}{2}} d\zeta_2 \ldots \int_0^{\zeta_{n-1}} {\zeta_n}^{\beta-\gamma-\frac{1}{2}} f(\zeta_n) \, d\zeta_n , \tag{2}$$

By Dirichlet technique Which yields

$$\int_0^z {\zeta_1}^{\beta-\gamma-\frac{1}{2}} d\zeta_1 \int_0^{\zeta_1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta) \, d\zeta = \int_0^z \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta) \, d\zeta \int_{\zeta}^z {\zeta_1}^{\beta-\gamma-\frac{1}{2}} d\zeta_1$$

$$= \int_0^z \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta) \, d\zeta \left[\frac{{\zeta_1}^{\beta-\gamma+\frac{1}{2}}}{\beta-\gamma+\frac{1}{2}}\right]_{\zeta}^{z}$$

$$= \frac{1}{\beta-\gamma+\frac{1}{2}} \int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right) \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta) \, d\zeta$$

Repeating the above step n–1 times, we obtain

$$\int_0^z {\zeta_1}^{\beta-\gamma-\frac{1}{2}} d\zeta_1 \int_0^{\zeta_1} {\zeta_2}^{\beta-\gamma-\frac{1}{2}} d\zeta_2 \ldots \int_0^{\zeta_{n-1}} {\zeta_n}^{\beta-\gamma-\frac{1}{2}} f(\zeta_n) \, d\zeta_n$$

$$= \frac{(\beta-\gamma+\frac{1}{2})^{1-n}}{(n-1)!} \int_0^z \; z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}})^{n-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta) \, d\zeta ,$$

which implies the fractional operator (for n=α),

$$I_z^{\alpha,\beta,\gamma} f(z) = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta) \, d\zeta , \tag{3}$$

where β,γ are real numbers and the function f(z) is analytic in simply connected region of the complex z-plane $\mathbb{C}$containing the origin, and the multiplicity of$(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}})^{\alpha-1}$is removed by requiring log$(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}})$ to be real when$\left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)$ >0.

In this paper, we define above operator by called it the fractional Integral operator-(β,γ).

For 0<p<1 ,the Bergman space A^p is the set of functions f analytic in the unit disk U:={z :z∈ ℂ ; |z| <1} with $\|f\|_{A^p}^p < \infty$ where the norm is defined by $\|f\|_{A^p}^p = \frac{1}{\pi}\int_u |f(z)|^p \, dA < \infty, \; z \in U$

And dA is denoted Lebesgue are a measure.

Theorem (2.1): Let α> 0,0<p<∞ and β, γ ∈R. Then, the operator $I_z^{\alpha,\beta,\gamma}$ is bounded in A^p and

$$\left\|I_z^{\alpha,\beta,\gamma} f(z)\right\|_{A^p}^p \le C\|f\|_{A^p}^p,$$

where

$$C:=\int_0^1 \left|\frac{(\beta-\gamma+\frac{1}{2})^{1-\alpha}}{\Gamma(\alpha)}\left(1-S^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} S^{\beta-\gamma-\frac{1}{2}}\right|^p.$$

Proof: Assume that $f(z) \in A^p$. Then, we have

$$\left\|I_z^{\alpha,\beta,\gamma} f(z)\right\|_{A^p}^p = \frac{1}{\pi}\left|I_z^{\alpha,\beta,\gamma} f(z)\right|^p \, dA$$

$$=\frac{1}{\pi}\int_0^1 \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta)d\zeta\right|^p \, dA$$

$$=\frac{1}{\pi}\int_0^1 \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(1-\frac{\zeta^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}}\right)^{\alpha-1} z^{(\alpha-1)(\beta-\gamma+\frac{1}{2})} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta)d\zeta\right|^p \, dA.$$

Let s=$\frac{\zeta}{z}$ *, then we obtain*

$$\left\|I_z^{\alpha,\beta,\gamma} f(z)\right\|_{A^p}^p = \frac{1}{\pi}\int_0^1 \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z (1-s)^{\alpha-1} z^{(\alpha-1)\left(\beta-\gamma+\frac{1}{2}\right)} (zs)^{\beta-\gamma-\frac{1}{2}} f(zs)ds\right|^p \, dA$$

$$=\frac{1}{\pi}\int_0^1 \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\frac{\zeta}{z}} (1-s)^{\alpha-1} z^{(\alpha-1)(\beta-\gamma+\frac{1}{2})+\beta-\gamma+\frac{1}{2}} s^{\beta-\gamma-\frac{1}{2}} f(zs)ds\right|^p \, dA$$

$$\le \frac{1}{\pi}\int_0^1 \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_u (1-s)^{\alpha-1} s^{\beta-\gamma-\frac{1}{2}} f(zs)ds\right|^p \, dA$$

$$\le \frac{1}{\pi}\int_0^1 \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_u (1-s)^{\alpha-1} s^{\beta-\gamma-\frac{1}{2}} ds\right|^p \left(\frac{1}{\pi}|f(z)|^p \, dA\right)$$

$$=C\,\|f\|_{A^p}^p.$$

This completes the proof.

Now, we discuss the semigroup properties of the integral operator

Theorem (2.2): Let f be analytic in the unit disk. Then, operator (2) satisfies

$$I_z^{\alpha,\beta,\gamma} I_z^{\lambda,\beta,\gamma} f(z) = I_z^{\alpha+\lambda,\beta,\gamma} f(z), \qquad \alpha,\lambda > 0. \tag{4}$$

Proof: For function f by using Dirichlet technique yields

$$I_z^{\lambda,\beta,\gamma} f(z) = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} I_z^{\lambda,\beta,\gamma} f(\zeta)\, d\zeta$$

$$= \frac{(\beta-\gamma+\frac{1}{2})^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}}\right.$$

$$\left.-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}}\left(\frac{(\beta-\gamma+\frac{1}{2})^{1-\lambda}}{\Gamma(\lambda)}\int_0^z \left(\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\lambda-1} \epsilon^{\beta-\gamma-\frac{1}{2}} f(\epsilon)\right) d\epsilon\, d\zeta$$

$$=\frac{(\beta-\gamma+\frac{1}{2})^{(1-\alpha)+(1-\lambda)}}{\Gamma(\alpha)\Gamma(\lambda)}\int_0^z \epsilon^{\beta-\gamma-\frac{1}{2}}f(\epsilon)\left[\int_\epsilon^\zeta \left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\left(\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\lambda-1}\zeta^{\beta-\gamma-\frac{1}{2}}\right]d\zeta d\epsilon\,.$$

(5)

Let $s=\frac{\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}$, then we have

$$\int_\epsilon^\zeta \left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\left(\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\lambda-1}\zeta^{\beta-\gamma-\frac{1}{2}}d\zeta$$

$$=\int_\epsilon^\zeta \left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\left(\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\lambda-1}\left(\frac{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}\right)^{\alpha-1}\left(\frac{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}\right)^{\lambda-1}\zeta^{\beta-\gamma-\frac{1}{2}}d\zeta$$

$$=\int_\epsilon^\zeta \left(\frac{z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}\right)^{\alpha-1}\left(\frac{\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}\right)^{\lambda-1}\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-2}\zeta^{\beta-\gamma-\frac{1}{2}}d\zeta$$

$$=\int_\epsilon^\zeta \left(1-\frac{\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}\right)^{\alpha-1}\left(\frac{\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}}\right)^{\lambda-1}\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-2}\zeta^{\beta-\gamma-\frac{1}{2}}d\zeta$$

$$=\int_\epsilon^\zeta (1-s)^{\alpha-1}(s)^{\lambda-1}\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-2}\zeta^{\beta-\gamma-\frac{1}{2}}d\zeta$$

$$=\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-2}\int_\epsilon^\zeta (1-s)^{\alpha-1}(s)^{\lambda-1}\zeta^{\beta-\gamma-\frac{1}{2}}ds$$

$$=\frac{\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-2}}{\beta-\gamma+\frac{1}{2}}\int_\epsilon^\zeta (1-s)^{\alpha-1}(s)^{\lambda-1}\zeta^{\beta-\gamma+\frac{1}{2}}ds$$

$$=\frac{\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-2}}{\beta-\gamma+\frac{1}{2}}\int_\epsilon^\zeta (1-s)^{\alpha-1}(s)^{\lambda-1}\left(\zeta^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)ds$$

$$=\frac{\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-1}}{\beta-\gamma+\frac{1}{2}}\int_\epsilon^\zeta (1-s)^{\alpha-1}(s)^{\lambda-1}ds$$

$$=\frac{\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-1}}{\beta-\gamma+\frac{1}{2}}\frac{\Gamma(\alpha)\Gamma(\lambda)}{\Gamma(\alpha+\lambda)}. \qquad (6)$$

From (5) and(6) ,we have

$$I_z^{\alpha,\beta,\gamma}I_z^{\lambda,\beta,\gamma}f(z)=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{2-\alpha-\lambda}}{\Gamma(\alpha)\Gamma(\lambda)}\int_0^z\frac{\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-1}}{\beta-\gamma+\frac{1}{2}}\frac{\Gamma(\alpha)\Gamma(\lambda)}{\Gamma(\alpha+\lambda)}\epsilon^{\beta-\gamma-\frac{1}{2}}f(\epsilon)\,d\epsilon$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha-\lambda}}{\Gamma(\alpha+\lambda)}\int_0^z\left(z^{\beta-\gamma+\frac{1}{2}}-\epsilon^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha+\lambda-1}\epsilon^{\beta-\gamma-\frac{1}{2}}f(\epsilon)\,d\epsilon \qquad (7)$$

$$=I_z^{\alpha+\lambda,\beta,\gamma}f(z)\,.$$

Example(2.1): Let $f(z)=z^v$, $v\in R$, we find the fractional Integral-(β,γ) of this function as follows :

$$I_z^{\alpha,\beta,\gamma}z^v=\frac{(\beta-\gamma+\frac{1}{2})^{1-\alpha}}{\Gamma(\alpha)}\int_0^z\left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\zeta^{\beta-\gamma-\frac{1}{2}+v}d\zeta$$

$$=\frac{(\beta-\gamma+\frac{1}{2})^{1-\alpha}}{\Gamma(\alpha)}\int_0^z\left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\left(\frac{z^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}}\right)^{\alpha-1}\zeta^{\beta-\gamma-\frac{1}{2}+v}d\zeta$$

$$=\frac{(\beta-\gamma+\frac{1}{2})^{1-\alpha}}{\Gamma(\alpha)}\int_0^z\left(\frac{z^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}}-\frac{\zeta^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}}\right)^{\alpha-1}z^{\alpha-1(\beta-\gamma+\frac{1}{2})}\zeta^{\beta-\gamma-\frac{1}{2}+v}d\zeta$$

Let $s=\left(\frac{\zeta}{z}\right)^{\beta-\gamma+\frac{1}{2}}\rightarrow\zeta=z\,s^{\frac{1}{\beta-\gamma+\frac{1}{2}}}\rightarrow d\zeta=\frac{z}{\beta-\gamma+\frac{1}{2}}s^{\frac{\beta-\gamma-\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}}ds\,.$

Then

$$I_z^{\alpha,\beta,\gamma} z^v = \frac{(\beta-\gamma+\frac{1}{2})^{-\alpha}}{\Gamma(\alpha)} \int_0^z (1-s)^{\alpha-1} z^{(\alpha-1)(\beta-\gamma+\frac{1}{2})+\beta-\gamma-\frac{1}{2}+v+1} s^{\frac{v}{\beta-\gamma+\frac{1}{2}}} ds$$

$$= \frac{(\beta-\gamma+\frac{1}{2})^{-\alpha}}{\Gamma(\alpha)} \int_0^z (1-s)^{\alpha-1} z^{\alpha(\beta-\gamma+\frac{1}{2})+v} s^{\frac{v}{\beta-\gamma+\frac{1}{2}}+1-1} ds$$

$$= \frac{(\beta-\gamma+\frac{1}{2})^{-\alpha}}{\Gamma(\alpha)} z^{\alpha(\beta-\gamma+\frac{1}{2})+v} \int_0^1 (1-s)^{\alpha-1} s^{\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}-1} ds$$

$$= \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{-\alpha}}{\Gamma(\alpha)} z^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)+v} \beta\left(\alpha, \frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}\right)$$

$$= \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{-\alpha}}{\Gamma(\alpha)} z^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)+v} \frac{\Gamma(\alpha)\Gamma\left(\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(\alpha+\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}\right)}$$

$$= \frac{z^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)+v}}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}} \frac{\Gamma\left(\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(\alpha+\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}\right)},$$

whereβ is the Beta function.

3 Fractional differential operator-(β, γ)

Corresponding to the fractional Integral operator-(β, γ), we define the following fractional differential operator-(β, γ).

Definition(3.1): The fractional derivative-(β, γ) of order α is defined, for a function $f(z)$, by

$$D_z^{\alpha,\beta,\gamma} f(z) = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{\zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta)}{\left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha}} d\zeta \quad , \quad 0 < \alpha < 1 \quad , \qquad (8)$$

where β,γ are real number and the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}})^{-\alpha}$ is removed by requiring $\log(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}})$ to be real when$\left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right) >0$.

Example (3.1): Let $f(z) = z^v$, $v \in R$, we find the fractional derivative -(β, γ) of this function as follows:

$$D_z^{\alpha,\beta,\gamma} f(z) = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{\zeta^{\beta-\gamma-\frac{1}{2}+v}}{\left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha}} d\zeta.$$

Let s=$\left(\frac{\zeta}{z}\right)^{\beta-\gamma+\frac{1}{2}} \rightarrow \zeta = z\, s^{\frac{1}{\beta-\gamma+\frac{1}{2}}} \rightarrow d\zeta = \frac{z}{\beta-\gamma+\frac{1}{2}} s^{\frac{\beta-\gamma-\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}} ds$

$$D_z^{\alpha,\beta,\gamma} f(z) = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{-\alpha} \left(\frac{z^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}}\right)^{-\alpha} \zeta^{\beta-\gamma-\frac{1}{2}+v} d\zeta$$

$$D_z^{\alpha,\beta,\gamma} f(z) = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \left(\frac{z^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}} - \frac{\zeta^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}}\right)^{-\alpha} \left(z^{\beta-\gamma+\frac{1}{2}}\right)^{-\alpha} \left(z\, s^{\frac{1}{\beta-\gamma+\frac{1}{2}}}\right)^{\beta-\gamma-\frac{1}{2}+v} \frac{z}{\beta-\gamma+\frac{1}{2}} s^{\frac{\beta-\gamma-\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}} ds$$

$$= \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z (1-s)^{(1-\alpha)-1} \quad z^{(1-\alpha)(\beta-\gamma-\frac{1}{2})+v} s^{\frac{v}{\beta-\gamma+\frac{1}{2}}} ds$$

$$= \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \quad z^{(1-\alpha)(\beta-\gamma-\frac{1}{2})+v} \int_0^1 (1-s)^{(1-\alpha)-1} s^{\frac{v}{\beta-\gamma+\frac{1}{2}}+1-1} ds$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\ (1-\alpha)(\beta-\gamma-\frac{1}{2})+v\ z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+v-1}\int_0^1(1-s)^{(1-\alpha)-1}s^{\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}-1}ds$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}}{\Gamma(1-\alpha)}\ (1-\alpha)(\beta-\gamma-\frac{1}{2})+v\ z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+v-1}\beta\left(1-\alpha\,,\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}\right)$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}}{\Gamma(1-\alpha)}\ (1-\alpha)(\beta-\gamma-\frac{1}{2})+v\ z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+v-1}\frac{\Gamma(1-\alpha)\Gamma\left(\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(1-\alpha+\frac{v+\beta-\gamma+\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}\right)}$$

$$=\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+v\ z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+v-1}\frac{\Gamma\left(\frac{v}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{v}{\beta-\gamma+\frac{1}{2}}+1\right)}$$

$$==\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\left(1-\alpha+\frac{v}{\beta-\gamma+\frac{1}{2}}\right)\ z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+v-1}\left(\frac{v}{\beta-\gamma+\frac{1}{2}}\right)!}{\left(1-\alpha+\frac{v}{\beta-\gamma+\frac{1}{2}}\right)\left(\frac{v}{\beta-\gamma+\frac{1}{2}}-\alpha\right)!}$$

$$D_z^{\alpha,\beta,\gamma}f(z)=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\Gamma\left(\frac{v}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{v}{\beta-\gamma+\frac{1}{2}}\right)}\quad z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+v-1}.$$

Proposition (3.1): Let $f(z)$be analytic function in the unit disk U of the form

$$f(z)=\sum_{n=1}^{\infty}a_n z^n\ ,\ \ z\in U \tag{9}$$

Then

$$D_z^{\alpha,\beta,\gamma}I_z^{\lambda,\beta,\gamma}f(z)=I_z^{\alpha,\beta,\gamma}D_z^{\lambda,\beta,\gamma}f(z)=f(z)$$

4 Fractional complex transform

Fractional complex transform is important and useful methods for fractional calculus is proposed to convert fractional differential equations into ordinary differential equations so that all analytical methods devoted to advanced calculus can be easily applied to fractional calculus see [3,5] .We use some properties of the fractional operator-(β,γ). The Wave transformation

$$\zeta=Az^{\alpha}+Bw^{\beta}+\ldots, \tag{10}$$

isspecial case of the fractional complex transform

$$\zeta=Az^{\alpha\left(\beta-\gamma-\frac{1}{2}\right)}+Bw^{\beta\left(\beta-\gamma-\frac{1}{2}\right)}+\ldots \tag{11}$$

Suppose the fractional complex transform

$$D_z^{\alpha,\beta,\gamma}f(z)=\frac{\partial}{\partial\zeta}D_z^{\alpha,\beta,\gamma}\zeta\,,\qquad\zeta=z^{\alpha\left(\beta-\gamma-\frac{1}{2}\right)} \tag{12}$$

If $D_z^{\alpha,\beta,\gamma}f(z)=\frac{\partial^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}f}{\partial z^{\alpha\left(\beta-\gamma-\frac{1}{2}\right)}}$

$$\frac{\partial^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}f}{\partial z^{\alpha\left(\beta-\gamma-\frac{1}{2}\right)}}=\frac{\partial f}{\partial\zeta}\frac{\partial^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}\zeta}{\partial z^{\alpha\left(\beta-\gamma-\frac{1}{2}\right)}}$$

$$=\frac{\partial f}{\partial\zeta}\sigma_{\alpha,\beta,\gamma}\ , \tag{13}$$

where $\sigma_{\alpha,\beta,\gamma}$ is the fractal index, which is usually determined in terms of gamma functions.

Example (4.1):Let $\zeta=z^{\alpha\left(\beta-\gamma-\frac{1}{2}\right)}$ $and\ f=\zeta^n$,$n\neq 0$ then, we have

$$\frac{\partial^{\alpha(\beta-\gamma+\frac{1}{2})} f}{\partial z^{\alpha(\beta-\gamma-\frac{1}{2})}} = \frac{\partial f}{\partial \zeta} \frac{\partial^{\alpha(\beta-\gamma+\frac{1}{2})} \zeta^n}{\partial z^{\alpha(\beta-\gamma-\frac{1}{2})}}$$

$$= \frac{\sigma_{\alpha,\beta,\gamma} \Gamma\left(\frac{n\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)} \quad z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+n\alpha-1}.$$

$$=\frac{\partial f}{\partial \zeta} \sigma_{\alpha,\beta,\gamma}$$

$$=\frac{n\sigma_{\alpha,\beta,\gamma} \quad z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+n\alpha-1}}{\left(\beta-\gamma+\frac{1}{2}\right)}.$$

Therefore we have

$$\sigma_{\alpha,\beta,\gamma} = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha} \Gamma\left(\frac{n\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)}{n\Gamma\left(1-\alpha+\frac{n\alpha}{\beta-\gamma+\frac{1}{2}}\right)}.$$

5 Applications

5.1 Distortion inequalities involving fractional derivatives[7]:

We discuss some applications of the fractional operators-(β,γ) (3) and (8) in geometric function theory.

Let Σ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,, \quad z \in U \tag{14}$$

Also, let $S$and Kdenote the subclasses of Σ consisting of functions which are, respect tively, univalent and convex in U. It is well known that if the function $f(z)$ given by (14) is in the class S, then

$$a_n \le n \,, \quad n \in N \setminus \{1\}. \tag{15}$$

Equality holds for the Koebe function

$$f(z) = \frac{z}{(1-z)^2} \,, \quad z \in U$$

Moreover, if the function f(z) given by (14) is in the class K, then

$$a_n \le 1 \,, \quad n \in N. \tag{16}$$

Equality holds for the function

$$f(z) = \frac{z}{1-\acute{z}} \,, \qquad z \in U.$$

Now, we shall also make use of the Fox-Wright generalization $_q\Psi_p$ $[z]$ of the hypergeometric $_qF_p$ function defined by[3]

$$_q\Psi_p \begin{bmatrix} (\alpha_1, A_1), \dots (\alpha_q, A_q); \\ (\beta_1, B_1), \dots (\beta_p, B_p); \end{bmatrix} z = q\Psi p\left[(\alpha_j, A_j)_{1,q;} (\beta_j, B_j)_{1,p;z}\right]$$

$$=\sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1,nA_1)\dots\Gamma(\alpha_q,nA_q)z^n}{\Gamma(\beta_1,nB_1)\dots\Gamma(\beta_p,nB_p)n!}$$

$$=\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q}\Gamma(\alpha_j,nA_j)z^n}{\prod_{j=1}^{p}\Gamma(\beta_j,nB_j)n!}; \qquad \text{(fox-wright function)}$$

where $A_j > 0$ for all j=1…q ,$B_j > 0$ for all j=1…p and

$1 + \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j \ \geq 0$for suitable values |z| < 1 andα_j, β_j are complex paramrters.

Theorem (5.1.1): Let $f \in S$. Then,

$$\left|D_z^{\beta-\gamma+\frac{1}{2}} f(z)\right| = r^{(1-\alpha)\left(\beta-\gamma+\frac{1}{2}\right)} \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \begin{bmatrix} (2,1), \left(1+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}, \frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \\ \left(1-\alpha+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}, \frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \end{bmatrix} r \tag{17}$$

$$(r = |z| \,; z \in U; 0 < \alpha < 1),$$

where the equality holds true for the Koebe function.

Proof: Suppose that the function $f\,(z) \in S$ is given by (14). Then,

$$D_z^{\alpha,\beta,\gamma} f(z) = D_z^{\alpha,\beta,\gamma} \sum_{n=1}^{\infty} a_n z^n$$

$$=\sum_{n=1}^{\infty} a_n (D_z^{\alpha,\beta,\gamma} z^n)$$

$$=\sum_{n=1}^{\infty} \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \Gamma\left(\frac{n}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n}{\beta-\gamma+\frac{1}{2}}\right)} a_n \;\; z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+n-1}, \quad a_1 = 1$$

$$= z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)} \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} a_{n+1} \;\; z^n$$

$$\left|D_z^{\alpha,\beta,\gamma} f(z)\right| = \left| \; z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)} \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} a_{n+1} \;\; z^n \right|$$

$$\left(Note\,;\, |a_n| \leq n \quad \Rightarrow |a_{n+1}| \leq n+1\right)$$

$$\left|D_z^{\alpha,\beta,\gamma} f(z)\right| \leq \; r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)} \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} (n+1) \;\; r^n$$

$$= r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)} \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} \frac{(n+1)\, n!}{n!} \;\; r^n$$

$$\left|D_z^{\alpha,\beta,\gamma} f(z)\right| \leq \; r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)} \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\Gamma(2+n\,)\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} \frac{r^n}{n!}$$

$$= r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)} \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \left[\begin{matrix} (2,1), \left(1+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}, \frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \\ \left(1-\alpha+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}, \frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \end{matrix} r\right]$$

$$= r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)} \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \quad {}_2\Psi_1\;[r]\,.$$

Theorem (5.1.2): Let $f \in S$. Then,

$$\left|D_z^{\alpha+k,\beta,\gamma} f(z)\right| \leq \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma+\frac{1}{2}\right)}} \left[\begin{matrix} (2,1), \left(1+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}, \frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \\ \left(\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}-(\alpha+k), \frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \end{matrix} r\right] \tag{18}$$

$$(r = |z|\,; z \in U; o < \alpha < 1, k \; is \; enteger \; number),$$

where the equality holds true for the Koebe function.

Proof: Suppose that the function $f\,(z) \in S$is given by (14). Then,

$$D_z^{\alpha+k,\beta,\gamma} f(z) = D_z^{\alpha+k,\beta,\gamma} \sum_{n=1}^{\infty} a_n z^n$$

$$=\sum_{n=1}^{\infty} a_n (D_z^{\alpha+k,\beta,\gamma} z^n)$$

$$=\sum_{n=1}^{\infty}\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}\Gamma\left(\frac{n}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n}{\beta-\gamma+\frac{1}{2}}\right)}a_n\ \ z^{(1-(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)+n-1},\quad a_1=1$$

$$=\ z^{-(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)}a_{n+1}\ \ z^n$$

$$\left(Note\ ; |a_n|\le n\ \ \Rightarrow |a_{n+1}|\le n+1\right)$$

$$D_z^{\alpha+k,\beta,\gamma}f(z)=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{z^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)}a_{n+1}\ \ z^n$$

$$\left|D_z^{\alpha+k,\beta,\gamma}f(z)\right|=\left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{z^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)}a_{n+1}\ \ z^n\right|$$

$$\le\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)}(n+1)\ \ r^n$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}}\sum_{n=0}^{\infty}\frac{\Gamma(n+2)\quad\Gamma\left(\frac{n}{\beta-\gamma+\frac{1}{2}}+\frac{1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n}{\beta-\gamma+\frac{1}{2}}+\frac{1}{\beta-\gamma+\frac{1}{2}}\right)}\frac{r^n}{n!}$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}}\left[\begin{matrix}(2,1),\left(1+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)},\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right);\\ \left(\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}-(\alpha+k),\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right);\end{matrix}\ r\right]$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}}\quad {}_2\Psi_1\ [r]\,.$$

Theorem (5.1.3): Let $f\ \in\ K$. Then,

$$\left|D_z^{\beta-\gamma+\frac{1}{2}}f(z)\right|=r^{(1-\alpha)\left(\beta-\gamma+\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\left[\begin{matrix}(1,1),\left(1+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)},\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right);\\ \left(1-\alpha+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)},\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right);\end{matrix}\ r\right]\qquad(19)$$

$$(r=|z|\ ; z\in U; o<\alpha<1),$$

where the equality holds true for the Koebe function.

Proof: Suppose that the function $f\ (z)\ \in\ K$ is given by (14). Then,

$$D_z^{\alpha,\beta,\gamma}f(z)=D_z^{\alpha,\beta,\gamma}\sum_{n=1}^{\infty}a_nz^n$$

$$=\sum_{n=1}^{\infty}a_n\left(D_z^{\alpha,\beta,\gamma}z^n\right)$$

$$=\sum_{n=1}^{\infty}\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\Gamma\left(\frac{n}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n}{\beta-\gamma+\frac{1}{2}}\right)}a_n\ \ z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)+n-1},\quad a_1=1$$

$$=\ z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\sum_{n=2}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)}a_{n+1}\ \ z^n$$

ISSN 2347-1921

$$\left|D_z^{\alpha,\beta,\gamma} f(z)\right| = \left| z^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} a_{n+1} \quad z^n\right|$$

$$\left(Note\,; |a_n| \le 1 \quad \Rightarrow |a_{n+1}| \le 1\right)$$

$$\left|D_z^{\alpha,\beta,\gamma} f(z)\right| \le \; r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} \quad r^n$$

$$= r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)}\frac{n!}{n!} \quad r^n$$

$$= r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\,\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)}\frac{r^n}{n!}$$

$$= r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1}\left[\begin{matrix}(1,1),\left(1+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)},\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \\ \left(1-\alpha+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)},\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right);\end{matrix}\; r\right]$$

$$= r^{(1-\alpha)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \quad {}_2\Psi_1\ [r]\,.$$

Theorem (5.1.4): Let $f \in k$. Then,

$$\left|D_z^{\alpha+k,\beta,\gamma} f(z)\right| \le \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma+\frac{1}{2}\right)}}\left[\begin{matrix}(2,1),\left(1+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)},\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \\ \left(\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}-(\alpha+k),\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right);\end{matrix}\; r\right] \qquad (20)$$

$$(r = |z|\,; z \in U; o < \alpha < 1, k \text{ is enteger number}),$$

where the equality holds true for the Koebe function.

Proof: Suppose that the function $f\,(z) \in S$is given by (14). Then,

$$D_z^{\alpha+k,\beta,\gamma} f(z) = D_z^{\alpha+k,\beta,\gamma}\sum_{n=1}^{\infty} a_n z^n$$

$$=\sum_{n=1}^{\infty} a_n\left(D_z^{\alpha+k,\beta,\gamma} z^n\right)$$

$$=\sum_{n=1}^{\infty}\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}\Gamma\left(\frac{n}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n}{\beta-\gamma+\frac{1}{2}}\right)} a_n \quad z^{(1-(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)+n-1}, \quad a_1 = 1$$

$$= z^{-(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} a_{n+1} \quad z^n$$

$$\left(Note\,; |a_n| \le 1 \quad \Rightarrow |a_{n+1}| \le 1\right)$$

$$D_z^{\alpha+k,\beta,\gamma} f(z) = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{z^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} a_{n+1} \quad z^n$$

$$\left|D_z^{\alpha+k,\beta,\gamma} f(z)\right| = \left| \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{z^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} a_{n+1} \ z^n \right|$$

$$\leq \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n+1}{\beta-\gamma+\frac{1}{2}}\right)} \quad r^n$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1) \quad \Gamma\left(\frac{n}{\beta-\gamma+\frac{1}{2}}+\frac{1}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-(\alpha+k)+\frac{n}{\beta-\gamma+\frac{1}{2}}+\frac{1}{\beta-\gamma+\frac{1}{2}}\right)} \frac{r^n}{n!}$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}} \left[\begin{matrix} (1,1), \left(1+\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}, \frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \\ \left(\frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}-(\alpha+k), \frac{1}{\left(\beta-\gamma+\frac{1}{2}\right)}\right); \end{matrix} \; r \right]$$

$$=\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{(\alpha+k)-1}}{r^{(\alpha+k)\left(\beta-\gamma-\frac{1}{2}\right)}} \quad {}_2\Psi_1 \ [r] \, .$$

5.2 Fractional differential equations

Now, we consider the Cauchy problem by employing the fractional differential operator-(β,γ).and determined the solution by terms of the fox-wright function.

Example(5.2.2): Consider the Cauchy problem in terms of the fractional differential operator-(β,γ) defined by (8)

$$D_z^{\alpha,\beta,\gamma} u(z) = F\big(z, u(z)\big)\,, \tag{21}$$

where $F\big(z,u(z)\big)$ is analytic in $u$and $u(z)$ is analytic in the unit disk. Thus, Fcan be expressed by

$$F(z,u) = \theta u(z)$$

Suppose the fractional complex transform

$$Z = z^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}$$

Then the solution can be expressed it as:

$$u(Z) = \sum_{k=0}^{\infty} a_k Z^k \tag{22}$$

Where $a_k$are constants.Substituting (22) into Eqution (21) implies

$$D_z^{\alpha,\beta,\gamma} \sum_{k=0}^{\infty} a_k Z^k - \theta \sum_{k=0}^{\infty} a_k Z^k$$

$$\frac{\partial}{\partial Z} \sum_{k=0}^{\infty} \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha-1} \Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right)} a_k \, Z^k - \theta \sum_{k=0}^{\infty} a_k Z^k = 0 \,. \tag{23}$$

Since $\sigma_{\alpha,\beta,\gamma} = \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha} \Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)}{k\Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right)}$.

Then we have

$$\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha} \Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)}{\Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right)} a_k - \theta a_{k-1} = 0;$$

$$a_k = \frac{\Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right) \quad \theta a_{k-1}}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha} \Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)}.$$

$for\ k = 0,1,2,\ldots$

$$a_0 = \frac{\Gamma(1-\alpha) \quad \theta a_{-1}}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}},$$

$$a_1 = \frac{\Gamma\left(1-\alpha+\frac{\alpha}{\beta-\gamma+\frac{1}{2}}\right) \quad \theta a_0}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha} \Gamma\left(\frac{\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)},$$

$$a_2 = \frac{\Gamma\left(1-\alpha+\frac{2\alpha}{\beta-\gamma+\frac{1}{2}}\right) \quad \theta a_1}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha} \Gamma\left(\frac{2\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)},$$

$$a_3 = \frac{\Gamma\left(1-\alpha+\frac{3\alpha}{\beta-\gamma+\frac{1}{2}}\right) \quad \theta a_{3-1}}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha} \Gamma\left(\frac{3\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)}.$$

$$\vdots$$

Then

$$a_k = \left(\frac{\theta}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}}\right)^{k} \frac{\Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right) \Gamma\left(1-\alpha+\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right) \Gamma\left(\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)}.$$

Thus

$$u(Z) = \sum_{k=0}^{\infty} \left(\frac{\theta}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}}\right)^{k} \frac{\Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right) \Gamma\left(1-\alpha+\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right) \Gamma\left(\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)} Z^k,$$

which is equivalent to

$$u(Z) = \sum_{k=0}^{\infty} \left(\frac{\theta}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}}\right)^{k} \frac{\Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right) \Gamma\left(1-\alpha+\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right) \Gamma\left(\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)} \frac{k!}{k!} Z^k$$

$$= \sum_{k=0}^{\infty} \left(\frac{\theta}{\left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}}\right)^{k} \frac{\Gamma(k+1) \quad \Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right) \Gamma\left(1-\alpha+\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right) \Gamma\left(\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)} \frac{Z^k}{k!}.$$

Since $\theta$is an arbitrary constant, we assume that

$\theta = \left(\beta-\gamma+\frac{1}{2}\right)^{\alpha}.$

Then

$$u(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1) \quad \Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right) \Gamma\left(1-\alpha+\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right) \Gamma\left(\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)} \frac{Z^k}{k!}$$

$$u(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+1) \quad \Gamma\left(1-\alpha+\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}\right) \Gamma\left(1-\alpha+\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}\right)}{\Gamma\left(\frac{k\alpha}{\beta-\gamma+\frac{1}{2}}+1\right) \Gamma\left(\frac{(k-1)\alpha}{\beta-\gamma+\frac{1}{2}}+1\right)} \frac{z^{k\left(\beta-\gamma+\frac{1}{2}\right)}}{k!}$$

$$= {}_3\Psi_2\left[\begin{matrix}(1,1),\left(1-\alpha-\frac{\alpha}{\left(\beta-\gamma+\frac{1}{2}\right)},\frac{\alpha}{\left(\beta-\gamma+\frac{1}{2}\right)}\right),\left(1-\alpha,\frac{\alpha}{\left(\beta-\gamma+\frac{1}{2}\right)}\right);\\ \left(1,\frac{\alpha}{\left(\beta-\gamma+\frac{1}{2}\right)}\right),\left(1-\frac{\alpha}{\left(\beta-\gamma+\frac{1}{2}\right)},\frac{\alpha}{\left(\beta-\gamma+\frac{1}{2}\right)}\right);\end{matrix}\; r\right].$$

Now, we study the (existence and uniqueness) of univalent solution[2,3], for the fractional differential equation (21),subject to the initial condition $u(0) = 0$, where $u : U \to C$ is an analytic function for all $z \in U$, and $f : U \times C \to C$ is an analytic function in $z \in U$. Let B represent complex Banach space of analytic functions in the unit disk.

Theorem(5.2.5):(Existence) Let the function $f : U \times C \to C$ be analytic such that $\|f\| \le M;\ M \ge 0$. Then, there exists a function $u : U \to C$ solving the problem(21).

Proof: Define the set $S : \{u \in B : \|u\| \le r, r > 0\}$, and the operator $P : S \to S$ by

$$Pu(z) := \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\ \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta,u(\zeta))\,d\zeta\,,\quad 0<\alpha<1.\ (22)$$

First , we show that P is bounded operator

$$|Pu(z)| := \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\ \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta,u(\zeta))\,d\zeta\right|$$

$$|Pu(z)| := \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\left|\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\ \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta,u(\zeta))\,d\zeta\right|$$

$$\le \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left|\left(z^{\beta-\gamma+\frac{1}{2}}-\zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\ \zeta^{\beta-\gamma-\frac{1}{2}} f\right| d\zeta$$

$$= \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left|\left(\frac{z^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}}-\frac{\zeta^{\beta-\gamma+\frac{1}{2}}}{z^{\beta-\gamma+\frac{1}{2}}}\right)^{\alpha-1}\left(z^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\ \zeta^{\beta-\gamma-\frac{1}{2}}\right| d\zeta.$$

Let $S = \left(\frac{\zeta}{z}\right)^{\beta-\gamma+\frac{1}{2}} \quad \to \zeta = z\, s^{\frac{1}{\beta-\gamma+\frac{1}{2}}} \qquad \to d\zeta = \frac{z}{\beta-\gamma+\frac{1}{2}} s^{\frac{\gamma-\beta-\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}}\ ds$

$$= \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left|(1-S)^{\alpha-1}\left(z^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\ \zeta^{\beta-\gamma-\frac{1}{2}}\right| d\zeta$$

$$|Pu(z)| \le \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left|(1-S)^{\alpha-1}\left(z^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1}\left(z\, s^{\frac{1}{\beta-\gamma+\frac{1}{2}}}\right)^{\beta-\gamma-\frac{1}{2}}\right| \frac{z}{\beta-\gamma+\frac{1}{2}} s^{\frac{\gamma-\beta-\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}}\ ds$$

$$= \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{-\alpha}}{\Gamma(\alpha)}\int_0^z \left|(1-S)^{\alpha-1}(z)^{\alpha-1\left(\beta-\gamma+\frac{1}{2}\right)+\beta-\gamma-\frac{1}{2}+1}\left(s^{\frac{1}{\beta-\gamma+\frac{1}{2}}}\right)^{\beta-\gamma-\frac{1}{2}}\right| s^{\frac{\gamma-\beta-\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}}\ ds$$

$$= \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{-\alpha}}{\Gamma(\alpha)}\int_0^z \left|(1-S)^{\alpha-1}(z)^{\alpha-1\left(\beta-\gamma+\frac{1}{2}\right)+\beta-\gamma-\frac{1}{2}+1}\ S^{\left(\beta-\gamma-\frac{1}{2}\right)\left(\frac{1}{\beta-\gamma+\frac{1}{2}}\right)+\frac{\gamma-\beta-\frac{1}{2}}{\beta-\gamma+\frac{1}{2}}}\right|\ ds$$

$$= \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{-\alpha}}{\Gamma(\alpha)}\int_0^1 (1-S)^{\alpha-1}S^0\quad ds$$

$$= \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{-\alpha}}{\Gamma(\alpha)}\int_0^1 (1-S)^{\alpha-1}S^{1-1}\quad ds.$$

Then

$$|Pu(z)| \le \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{-\alpha}}{\Gamma(\alpha)}\beta(\alpha,1)$$

ISSN 2347-1921

:=r,

that is $\|Pu\|_B = sup_{z\in U}|(Pu)(z)|$. We proceed to prove that $P: S \to S$ is continuousoperator. Since f is continuous function on $U \times S$, then it is uniformly continuous on a compact set $\breve{U} \times S$, *where*

$$\breve{U} := \{z \in U: |z| \le l, l \in (0,1)\}.$$

Hence, given $\epsilon > 0, \exists \delta > 0$ such that for all $u, v \in S$ we have

$\|f(z,u) - f(z,v)\| < \frac{\epsilon\Gamma(\alpha)}{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha} l^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}\beta(1,\alpha)}$, for all $\|u - v\| < \delta$

$$|(Pu)(z)) - (Pv)(z)| = \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, u(\zeta))\, d\zeta - \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, v(\zeta))\, d\zeta\right|$$

$$= \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\left|\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, u(\zeta))\, d\zeta - \int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, v(\zeta))\, d\zeta\right|$$

$$\le \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\left|\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, u(\zeta))\, d\zeta\right| - \left|\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, v(\zeta))\, d\zeta\right|$$

$$\le \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left|\left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, u(\zeta))\right| d\zeta - \left|\int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, v(\zeta))\right| d\zeta$$

$$= \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left|\left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}}\right| \times \left|f(\zeta, u(\zeta)) - f(\zeta, v(\zeta))\right| d\zeta.$$

Then

$$|(Pu)(z)) - (Pv)(z)| \le \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha} l^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}\beta(1,\alpha)}{\Gamma(\alpha)} \times \frac{\epsilon\Gamma(\alpha)}{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha} l^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}\beta(1,\alpha)}$$

$$=\epsilon.$$

Thus, P is a continuous mapping on S. Now, we show that P is an equicontinuous mapping *on* S. *For* $z_1, z_2 \in \breve{U}$such that $z_1 \neq z_2$, then for all $u \in S$ we obtain

$$|(P(u)(z_1) - P(u)(z_2)| = \left|\frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left({z_1}^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, u(\zeta))\, d\zeta - \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\int_0^z \left({z_2}^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, v(\zeta))\, d\zeta\right|$$

$$\le \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\left(\int_0^z \left|\left({z_1}^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, u(\zeta))\right| d\zeta + \int_0^z \left|\left({z_2}^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, v(\zeta))\right| d\zeta\right)$$

$$\le \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\left(\int_0^z \left|\left({z_1}^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}}\right| d\zeta + \int_0^z \left|\left({z_2}^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}}\right| d\zeta\right)$$

$$\le \frac{M\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)}\left(l^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}\beta(1,\alpha) + l^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}\beta(1,\alpha)\right)$$

$$\leq \frac{2M\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha} l^{\alpha\left(\beta-\gamma+\frac{1}{2}\right)}}{\Gamma(\alpha)} \beta(1,\alpha),$$

which is independent on u. Hence, P is an equicontinuous mapping on S. The(Arzela-Ascoli theorem) yields that every sequence of functions from $P(S)$ has got a uniformly convergent subsequence, and therefore $P(S)$ is relatively compact. (Schauder's fixed point theorem) asserts that P has a fixed point. By construction, a fixed point of P is a solution of the initial value problem (21).

Theorem(5.2.6): (Uniqueness) Let the function f be bounded and fulfill a Lipschitz condition with respect to the second variable: i.e.,

$$\|f(z,u) - f(z,v)\| \leq L\|u - v\|,$$

for some $L > 0$ independent of u, v and z.

If $\frac{L\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)} \beta(1,\alpha) < 1$, then there exist a unique function $u : U \to C$ solving the initial value problem (21).

Proof: We need only to prove that the operator P in Eq.(22) has a unique fixed point.

$$|(Pu)(z)) - (Pv)(z)|$$

$$= \left| \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, u(\zeta))\, d\zeta - \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} f(\zeta, v(\zeta))\, d\zeta \right|$$

$$\leq \frac{\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \left| \left(z^{\beta-\gamma+\frac{1}{2}} - \zeta^{\beta-\gamma+\frac{1}{2}}\right)^{\alpha-1} \zeta^{\beta-\gamma-\frac{1}{2}} \right| \times \left| f(\zeta, u(\zeta)) - f(\zeta, v(\zeta)) \right| d\zeta$$

$$\leq \frac{L\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)} \beta(1,\alpha) \,.\, \|u - v\| \;.$$

Then, for all u, v, we obtain

$$\|Pu - Pv\| \frac{L\left(\beta-\gamma+\frac{1}{2}\right)^{1-\alpha}}{\Gamma(\alpha)} \beta(1,\alpha) \,.\, \|u - v\|.$$

Thus, the operator P is a contraction mapping then in view of (Banach fixed point theorem), P has a unique fixed point which corresponds to the solution of the initial value problem (21).

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