

# Properties of The Function $\phi$ and Composition Operator $C_\phi$ Induced by The Function $\phi$

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## Abstract

Let  $U$  denote the unit ball in the complex plane, the Hardy space  $H^2$  is the set of functions  $f(z) = \sum_{n=0}^{\infty} f^{(n)} z^n$  holomorphic on  $U$  such that  $\sum_{n=0}^{\infty} |f^{(n)}|^2 < \infty$  with  $f^{(n)}$  denotes then the Taylor coefficient of  $f$ .

Let  $\psi$  be a holomorphic self-map of  $U$ , the composition operator  $C_\psi$  induced by  $\psi$  is defined on  $H^2$  by the equation

$$C_\psi f = f \circ \psi \quad (f \in H^2)$$

In this paper we have studied the composition operator induced by the automorphism  $\phi$  and discussed the adjoint of the composition of the symbol  $\phi$ . We have look also for some known properties on composition operators and tried to see the analogue properties in order to show how the results are changed by changing the function  $\psi$  in  $U$ .

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties.

## Introduction

This paper consists of two sections. In section one, we are going to the automorphism  $\phi$  and properties of  $\phi$ , and also discuss the interior and exterior fixed points of  $\phi$  and also discuss  $\phi$  is a rotation around the origin and  $\phi$  is elliptic and  $\phi$  is a linear fractional transformation.

In section two, we are going to the Composition Operator  $C_\phi$  induced by the symbol  $\phi$  and properties of  $C_\phi$ , and also discuss the adjoint of Composition Operator  $C_\phi$  induced by the symbol  $\phi$  and also discuss  $C_\phi$  is an invertible operator and  $C_\phi$  is a unitary operator and define eigenvalue of  $C_\phi$

## Section One

### **Properties of The Function $\phi$**

In section one ,we are going to the automorphism  $\phi$  and properties of  $\phi$  ,and also discuss the interior and exterior fixed points of  $\phi$  and also discuss  $\phi$  is a rotation around the origin and  $\phi$  is elliptic and  $\phi$  is a linear fractional transformation .

#### **Definition(1.1)** : [4]

Let  $U = \{z \in \mathbb{C} : |z| < 1\}$  is called unit ball in complex numbers  $\mathbb{C}$  and  $\partial U = \{z \in \mathbb{C} : |z| = 1\}$  is called boundary of  $U$

#### **Definition (1.2):**

For  $\beta \in U$ , define  $\phi(z) = \frac{-3z}{3-6\bar{\beta}z}$  ( $z \in U$ ). Since the denominator equal zero only at  $z = \frac{1}{2\bar{\beta}}$ , the function  $\phi$  is holomorphic on the ball  $\{z \in \mathbb{C} : |z| < \frac{1}{2|\beta|}\}$ . Since  $\beta \in U$ , then this ball contains  $U$ . Hence  $\phi$  take  $U$  into  $U$  and holomorphic on  $U$ .

#### **Definition(1.3)** : [8]

Let  $\psi : U \rightarrow U$  be holomorphic map on  $U$ . We say that  $\psi$  is called conformal automorphism or automorphism of  $U$  if and only if  $\psi$  is injective and surjective .

#### **Remark (1.4)** :

If  $\beta \in U$ , then  $\phi$  is conformal automorphism or automorphism of  $U$ .

#### **Definition(1.5)** : [8]

A point  $p \in \mathbb{C}$  is a fixed point for the function  $\psi$  , if  $\psi(p) = p$ .

#### **Proposition (1.6)** :

If  $\beta \in U$ , then  $0, \frac{1}{\bar{\beta}}$  are fixed points for  $\phi$ .

#### **Proof** :

Let  $\phi(z) = z$ , that is  $\frac{-3z}{3-6\bar{\beta}z} = z$ , therefore  $6\bar{\beta}z^2 - 6z = 0$ . Hence  $\phi$  has two fixed points  $z_1 = 0, z_2 = \frac{1}{\bar{\beta}}$

**Definition(1.7):** [4]

Let  $\psi : U \rightarrow U$  be holomorphic map on  $U$  that fixed point  $r$ , then:

- 1)  $r$  is interior fixed point for  $\psi$  if  $r \in U$
- 2)  $r$  is exterior fixed point for  $\psi$  if  $r \notin U$

**Proposition (1.8):**

If  $\phi(z) = \frac{-3z}{3-6\beta z}$ , then 0 is interior fixed point and  $\frac{1}{\beta}$  is exterior fixed point for  $\phi$ .

**Proof :**

Since  $\phi$  has two fixed points  $z_1 = 0$ ,  $z_2 = \frac{1}{\beta}$ ,  $|z_1| = |0| = 0 < 1$ . Thus  $z_1$  is interior fixed point.

Since  $\beta \in U$ , then  $|\beta| < 1$  and  $|\bar{\beta}| = |\beta| < 1$ , therefore  $\left|\frac{1}{\bar{\beta}}\right| = \frac{1}{|\beta|} > 1$ , hence  $|z_2| = \left|\frac{1}{\bar{\beta}}\right| > 1$ . Thus  $z_2$  is exterior fixed point.

**Remark(1.9) :**

1- For  $\beta \in U$ ,  $\phi^{-1}(z) = \frac{-3z}{3-6\bar{\beta}z} = \phi(z)$

2- For  $\beta \in U$ ,  $\phi'(0) = -1$ ,  $\phi'(\beta) = \frac{-9}{(3-6|\beta|^2)^2}$ .

**Definition(1.10):** [11]

If  $\psi : U \rightarrow U$  is holomorphic map on  $U$ . We say that  $\psi$  is a rotation around the origin if there exist  $\lambda \in \partial U$  such that  $\psi(z) = \lambda z$  ( $z \in U$ )

**Proposition (1.11):**

If  $\beta = 0$ ,  $\phi$  is a rotation around the origin

**Proof:**

Since  $\phi(z) = \frac{-3z}{3-6\beta z}$ , since  $\beta = 0$ , then  $\phi(z) = -z = \sigma z$ ,  $\sigma = -1 \in \partial U$ , hence  $\phi$  is a rotation around the origin.

**Theorem (1.12) :** [11]

If  $\psi : U \rightarrow U$  is holomorphic map on  $U$ , then  $\psi$  is elliptic if and only if  $\psi$  is automorphism that has an interior fixed point.

**Proposition (1.13) :**

If  $\beta \in U$ , then  $\phi$  is elliptic

**Proof :**

From (1.4),  $\phi$  is automorphism, and from (1.8)  $\phi$  has an interior fixed point, hence  $\phi$  is elliptic.

**Definition(1.14):** [10]

A linear fractional transformation is a mapping of the form  $\tau(z) = \frac{az+b}{cz+d}$ , where  $a, b, c$ , and  $d$  are complex numbers, and sometime denote it by  $\tau_A(z)$  where  $A$  is the non-singular  $2 \times 2$  complex matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

**Proposition (1.15) :**

$\phi$  is a linear fractional transformation.

**Proof :**

Since  $\phi(z) = \frac{-3z}{3-6\bar{\beta}z} = \frac{az+b}{cz+d}$  such that  $a = -3, b = 0, c = -6\bar{\beta}, d = 3$  and  $a, b, c$ , and  $d$  are complex numbers and  $A = \begin{bmatrix} -3 & 0 \\ -6\bar{\beta} & 3 \end{bmatrix}$ , hence by (1.14)  $\phi$  is a linear fractional transformation.

**Section Two****The composition Operator  $C_\phi$  Induced by The Function  $\phi$** 

In section two, we are going to the Composition Operator  $C_\phi$  induced by the symbol  $\phi$  and properties of  $C_\phi$ , and also discuss the adjoint of Composition Operator  $C_\phi$  induced by the symbol  $\phi$  and also discuss  $C_\phi$  is an invertible operator and  $C_\phi$  is a unitary operator and define eigenvalue of  $C_\phi$

**Definition(2.1):** [4]

Let  $U$  denote the unit ball in the complex plane, the Hardy space  $H^2$  is the set of functions  $f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n)z^n$  holomorphic on  $U$  such that  $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$  with  $f^{\wedge}(n)$  denotes then the Taylor coefficient of  $f$ .

**Remark (2.2) :** [1]

We can define an inner product of the Hardy space functions as follows:

$f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n) z^n$  and  $g(z) = \sum_{n=0}^{\infty} g(n) z^n$ , then inner product of  $f$  and  $g$  is:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} f^{\wedge}(n) \overline{g^{\wedge}(n)}$$

**Definition (2.3) :**[10]

Let  $\alpha \in U$ , define  $k_{\alpha}(z) = \frac{1}{1 - \alpha z}$  ( $z \in U$ ). Since  $\alpha \in U$  then  $|\alpha| < 1$ , hence the geometric series

$$\sum_{n=0}^{\infty} |\alpha|^{2n} \text{ is convergent and thus } k_{\alpha} \in H^2 \text{ and } k_{\alpha}(z) = \overline{\alpha} z^n.$$

**Definition(2.4) :** [4]

Let  $\psi : U \rightarrow U$  and holomorphic on  $U$ , the composition operator  $C_{\psi}$  induced by  $\psi$  is defined on  $H^2$  by the equation  $C_{\psi} f = f \circ \psi$  ( $f \in H^2$ )

**Definition(2.5) :** [2]

Let  $T$  be a bounded operator on a Hilbert space  $H$ , then the norm of an operator  $T$  is defined by  $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$ .

**Littlewood's Subordination principle (2.6) :** [11]

Let  $\psi : U \rightarrow U$  and holomorphic on  $U$  with  $\psi(0) = 0$ , then for each  $f \in H^2$ ,  $f \circ \psi \in H^2$  and  $\|f \circ \psi\| \leq \|f\|$ . The goal of this theorem  $C_{\psi} : H^2 \rightarrow H^2$ .

**Proposition(2.7) :**

If  $\phi(z) = \frac{-3z}{3-6\beta z}$ , then for each  $f \in H^2$ ,  $f \circ \phi \in H^2$  and  $\|f \circ \phi\| \leq \|f\|$

**Proof :**

Since  $\phi : U \rightarrow U$  and holomorphic on  $U$  with  $\phi(0) = 0$  by (1 . 6), then by (2 . 6)

$f \in H^2$ ,  $f \circ \phi \in H^2$  and  $\|f \circ \phi\| \leq \|f\|$ , hence  $C_{\phi} : H^2 \rightarrow H^2$

**Remark (2.8) :** [4]

1) One can easily show that  $C_{\kappa}C_{\psi} = C_{\psi \circ \kappa}$  and hence  $C_{\psi}^n = C_{\psi}C_{\psi} \cdots C_{\psi}$   
 $= C_{\psi \circ \psi \circ \cdots \circ \psi} = C_{\psi_n}$

2)  $C_{\psi}$  is the identity operator on  $H^2$  if and only if  $\psi$  is identity map from  $U$  into  $U$  and holomorphic on  $U$ .

3) It is simple to prove that  $C_{\kappa} = C_{\psi}$  if and only if  $\kappa = \psi$ .

**Theorem (2.9) :** [11]

Let  $\psi : U \rightarrow U$  and holomorphic on  $U$ .  $C_{\psi}$  is an invertible operator on  $H^2$  if and only if  $\psi$  automorphism of  $U$  and  $C_{\psi}^{-1} = C_{\psi^{-1}}$

**Proposition(2.10) :**

$C_{\phi}$  is an invertible operator on  $H^2$

**Proof :**

Since  $\phi$  is automorphism of  $U$  by (1. 4), hence  $C_{\phi}$  is an invertible operator on  $H^2$ .

**Theorem (2.11) :** [5]

$\bigvee_{\alpha \in U} \{K_{\alpha}\}$  forms a dense subset of  $H^2$ .

**Definition(2.12):** [12]

Let  $H^{\infty}$  be the set of all bounded holomorphic maps on  $U$ .

**Theorem (2.13) :** [10]

Let  $\psi : U \rightarrow U$  and holomorphic on  $U$ , then for all  $\alpha \in U$

$$C_{\psi}^* K_{\alpha} = K_{\psi(\alpha)}$$

**Definition(2.14):** [6]

Let  $g \in H^{\infty}$ , the Toeplits operator  $T_g$  is the operator on  $H^2$  given by :

$$(T_g f)(z) = g(z) f(z) \quad (f \in H^2, z \in U)$$

**Remark (2.15) :** [7]

For each  $f \in H^2$ , it is well- know that  $T_h^* f = T_{\bar{h}} f$ , such that  $h \in H^{\infty}$ .

**Proposition(2.16) :**

Let  $\beta \in U$ ,  $C_\phi^* = T_g C_\gamma T_h^*$  where  $h(z) = 1 - 2\bar{\beta}z$ ,  $g(z) = 1$ ,  $\gamma(z) = \frac{6\beta - 3z}{3}$

**Proof :**

By (2-17),  $T_h^* f = T_h f$  for each  $f \in H^2$ . Hence for all  $\alpha \in U$ ,

$$\langle T_h^* f, k_\alpha \rangle = \langle T_h f, k_\alpha \rangle = \langle f, T_h^* k_\alpha \rangle \cdots (2-1)$$

On the other hand ,

$$\langle T_h^* f, k_\alpha \rangle = \langle f, T_h f \rangle = \langle f, h(\alpha) k_\alpha \rangle \cdots (2-2)$$

From (2-1) and (2-2) one can see that  $T_h^* k_\alpha = h(\alpha) k_\alpha$ . Hence  $T_h^* k_\alpha = \overline{h(\alpha)} k_\alpha$ .

Calculation give:

$$\begin{aligned} C_\phi^* k_\alpha(z) &= k_{\phi(\alpha)}(z) \\ &= \frac{1}{1 - \overline{\phi_\beta(\alpha)} z} = \frac{1}{1 + \frac{3\alpha z}{3 - 6\beta\alpha}} \\ &= \frac{1}{\frac{3 - 6\beta\alpha + 3\alpha z}{3 - 6\beta\alpha}} = \frac{3 - 6\beta\alpha}{3 - \alpha(6\beta - 3z)} = \frac{\overline{(1 - 2\alpha\beta)}}{1 - \overline{\alpha} \left( \frac{6\beta - 3z}{3} \right)} \\ &= \overline{(1 - 2\alpha\beta)} \cdot (1) \cdot \frac{1}{1 - \overline{\alpha} \left( \frac{6\beta - 3z}{3} \right)} \\ &= \overline{h(\alpha)} \cdot T_g K_\alpha(\gamma(z)) = T_g \overline{h(\alpha)} K_\alpha(\gamma(z)) \\ &= T_g \overline{h(\alpha)} C_\gamma K_\alpha(z) = T_g C_\gamma \overline{h(\alpha)} K_\alpha(z) \\ &= T_g C_\gamma T_h^* K_\alpha(z), \text{ therefore} \end{aligned}$$

$$C_\phi^* k_\alpha(z) = T_g C_\gamma T_h^* k_\alpha(z).$$

But  $\overline{\bigcup_{\alpha \in U} \{K_\alpha\}} = H^2$ , then  $C_\phi^* = T_g C_\gamma T_h^*$

**Definition (2.17) : [3]**

Let  $T$  be an operator on a Hilbert space  $H$ ,  $T$  is called normal operator if  $T T^* = T^* T$ , and  $T$  is called unitary operator if  $T T^* = T^* T = I$ , and  $T$  is called isometric operator if  $T^* T = I$

**Theorem (2.18) :** [9]

If  $\psi : U \rightarrow U$  is holomorphic map on  $U$ , then  $C_\psi$  is normal if and only if  $\phi(z) = \lambda z$  for some  $\lambda, |\lambda| \leq 1$

**Theorem (2.19) :**

If  $\psi : U \rightarrow U$  be holomorphic map on  $U$ , then  $C_\psi$  is unitary if and only if  $\phi(z) = \lambda z$  for some  $\lambda, |\lambda| = 1$

**Proof :**

Suppose  $C_\psi$  is unitary, hence by (2.17)  $C_\psi C_\psi^* = C_\psi^* C_\psi = I$ , hence  $C_\psi C_\psi^* = C_\psi^* C_\psi$ , hence  $C_\psi$  is normal operator, hence by (2.18)  $\psi(z) = \lambda z$  for some  $\lambda, |\lambda| \leq 1$ . It is enough to show that  $|\lambda| = 1$

$$\begin{aligned} C_\psi^* C_\psi K_\beta(z) &= C_\psi^* K_{\psi(\beta)}(\psi(z)) = K_{\psi(\beta)}(\psi(z)). \\ &= \frac{1}{1 - \overline{\psi(\beta)} \psi(z)} = \frac{1}{1 - \bar{\lambda} \bar{\beta} \lambda z} = \frac{1}{1 - |\lambda|^2 \bar{\beta} z} \end{aligned}$$

On the other hand  $C_\psi^* C_\psi K_\beta(z) = K_\beta(z)$ , hence  $\frac{1}{1 - |\lambda|^2 \bar{\beta} z} = K_\beta(z) = \frac{1}{1 - \bar{\beta} z}$ .

Thus  $|\lambda|^2 \bar{\beta} = \bar{\beta}$ , then  $|\lambda| = 1$ .

Conversely, Suppose  $\psi(z) = \lambda z$  for some  $\lambda, |\lambda| = 1$ . For  $\beta \in U$ , for every  $z \in U$

$$\begin{aligned} C_\psi^* C_\psi K_\beta(z) &= C_\psi^* K_{\psi(\beta)}(\psi(z)) = K_{\psi(\beta)}(\psi(z)) \\ &= \frac{1}{1 - \overline{\psi(\beta)} \psi(z)} = \frac{1}{1 - \bar{\lambda} \bar{\beta} \lambda z} = \frac{1}{1 - |\lambda|^2 \bar{\beta} z} \\ &= \frac{1}{1 - \bar{\beta} z} = K_\beta(z) \end{aligned}$$

Moreover, for every  $z \in U$

$$\begin{aligned} C_\psi C_\psi^* K_\beta(z) &= C_\psi K_{\psi(\beta)}(z) = K_{\psi(\beta)}(\psi(z)) \\ &= \frac{1}{1 - \overline{\psi(\beta)} \psi(z)} = \frac{1}{1 - \bar{\lambda} \bar{\beta} \lambda z} = \frac{1}{1 - |\lambda|^2 \bar{\beta} z} \\ &= \frac{1}{1 - \bar{\beta} z} = K_\beta(z) \end{aligned}$$

hence  $C_\psi C_\psi^* = C_\psi^* C_\psi = I$  on the family  $\{K_\alpha\}_{\alpha \in U}$ . But by (2.11)  $\{K_\alpha\}_{\alpha \in U}$  forms a dense subset of  $H^2$ , hence

$C_\psi C_\psi^* = C_\psi^* C_\psi = I$  on  $H^2$ . Therefore  $C_\psi$  is unitary composition operator on  $H^2$ .

**Proposition(2.20) :**

If  $\beta = 0$ , then  $C_\phi$  is an unitary composition operator .

**Proof :**

Since  $\phi(z) = \frac{-3z}{3-6\beta z}$ , since  $\beta = 0$ ,  $\phi(z) = \frac{-3z}{3-6\beta z} = -z = \lambda z$ ,  $\lambda = -1$ ,  $|\lambda| = 1$  hence by (2.19)  $C_\phi$  is unitary composition operator .

**Definition (2.21) :** [12]

Let  $\psi : U \rightarrow U$  and holomorphic on  $U$ , the eigenvalue equation for the composition operator is define by  $C_\psi f = \kappa f$  or  $f \circ \psi = \kappa f$  .

**Theorem (2.22):** [11]

Let  $\psi : U \rightarrow U$  and holomorphic on  $U$ , and that fixes the point  $p \in U$  and suppose that  $C_\psi f = \kappa f$  for some non-constant  $f \in H^2$  and some  $\kappa \in \mathbb{C}$ . Then  $\kappa = (\psi'(p))^n$  for some  $n = 0, 1, 2, \dots$ .

**Proposition(2.23) :**

$(-1)^n$  is an eigenvalue of  $C_\phi$  for some  $n = 0, 1, 2, \dots$ .

**Proof :**

Since  $\phi(z) = \frac{-3z}{3-6\bar{\beta}z}$ ,  $\phi'(z) = \frac{(3-6\bar{\beta}z)(-3) - (-3z)(-6\bar{\beta})}{(3-6\bar{\beta}z)^2} = \frac{-9}{(3-6\bar{\beta}z)^2}$ , and since  $\phi$  fixed the point  $0 \in U$ , and by (2.22)  $\kappa = (\phi'(0))^n = (-1)^n$  is an eigenvalue of  $C_\phi$  for some  $n = 0, 1, 2, \dots$

**Definition (2.24) :** [7]

The spectrum of an operator  $T$  on a Hilbert space  $H$ , denoted by  $\sigma(T)$  is the set of all complex numbers  $\lambda$  for which  $T - \lambda I$  is not invertible . The spectral radius of  $T$ , denoted by  $r(T)$  is defined as  $r(T) = \sup \{|\lambda| : \lambda \in \sigma(T)\}$  .

**Theorem (2.25) :** [5]

If  $\psi : U \rightarrow U$  and holomorphic on  $U$ , and  $\psi$  has interior fixed point  $p$  then  $r(C_\psi) = 1$ .

**Proposition (2.26) :**

If  $\phi(z) = \frac{-3z}{3-6\bar{\beta}z}$ , then  $r(C_\phi) = 1$

**Proof :**

Since  $\phi : U \rightarrow U$  and holomorphic on  $U$ , and  $\phi$  has interior fixed point  $0$  . By (2.25)  $r(C_\phi) = 1$ .

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## خواص الدالة $\phi$ والمؤثر التركيبي $C_\phi$ المتولد بالدالة $\phi$

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كلية التربية

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### المستخلص

ليكن  $U$  يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي  $H^2$  هو مجموعة كل الدوال  $f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n) z^n$

التحليلية على  $U$  بحيث أن  $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$  ، يرمز إلى معاملات تيلر النونية.

لتكن  $\psi: U \rightarrow U$  دالة تحليلية على  $U$  ، المؤثر التركيبي المحتث من  $\psi$  يعرف على فضاء هاردي  $H^2$  بواسطة:

$$C_\psi f = f \circ \psi \quad (f \in H^2).$$

درسنا في هذا البحث المؤثر التركيبي المحتث من الدالة  $\phi$  حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المحتث من الدالة  $\phi$ .

بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير

الدالة التحليلية  $\psi$ .

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة

وكذلك برهنا بعض النتائج.