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ON A NEW CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. In the present paper, we define a new class $G(\alpha, \beta, b)(\alpha \geq 0; -1 \leq \beta \leq 0; b \in \mathbb{C})$ of functions which are analytic in the unit disk. A necessary and sufficient condition for functions to be in $G(\alpha, \beta, b)$ is obtained. Also for this class we get the radii of close-to-convexity, starlikeness, and convexity. Furthermore, we give an application involving fractional calculus for functions in $G(\alpha, \beta, b)$.

1. INTRODUCTION

Let W be the class of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0), \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. The class of functions $f(z) \in W$, which are starlike of order α and convex of order α $(0 \leq \alpha < 1)$ were investigated by Silverman [3].

Let $G(\alpha, \beta, b)$ denote the class of functions $f(z) \in W$ which satisfy the condition

$$Re\left\{\beta\frac{f(z)}{z} + (1-\beta)f'(z) + \alpha z f''(z)\right\} > 1 - |b| \tag{1.2}$$

for some α $(\alpha \geq 0), -1 \leq \beta \leq 0$ and $b \in \mathbb{C}$, and for all $z \in U$.

The class $G(\alpha, 0, 1-\gamma)$ was introduced by Altintas [1] who obtained several results concerning this class. The class $G(\alpha, 0, b)$ was introduced by Srivastava and Owa [6]. We give some properties of functions of $G(\alpha, \beta, b)$, radii of close-to-convexity, starlikeness, and convexity, and some distortion theorems involving fractional calculus.

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2. Interesting Properties of the Class $G(\alpha, \beta, b)$

Theorem 2.1. *A function $f(z) \in W$ is in the class $G(\alpha, \beta, b)$ if and only if*

$$\sum_{n=2}^{\infty}[\beta + n(1-\beta+\alpha n-\alpha)]a_n \leq |b|. \tag{2.1}$$

The result (2.1) is sharp.

Proof. Assume that $f(z) \in G(\alpha, \beta, b)$. Then we find from (1.2) that

$$Re\left\{\beta\left[1-\sum_{n=2}^{\infty} a_n z^{n-1}\right]+(1-\beta)\left[1-\sum_{n=2}^{\infty} n a_n z^{n-1}\right]+ \alpha z\left[-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right]\right\} > 1-|b|.$$

If we choose z to be the real and let $z \to 1^-$, we get $1-\sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]a_n \geq 1-|b|$, which is equivalent to (2.1). Conversely, assume that (2.1) is true. Then we have

$$\left|\beta\frac{f(z)}{z}-(1-\beta)f'(z)-\alpha z f''(z)-1\right| \leq \sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]a_n \leq |b|.$$

This implies that $f(z) \in G(\alpha, \beta, b)$. The result (2.1) is sharp for the function

$$f(z) = z - \frac{|b|}{\beta+n(1-\beta+\alpha n-\alpha)}z^n \ (n \geq 2). \tag{2.2}$$

□

Theorem 2.2. *If $f(z) \in G(\alpha, \beta, b)$, then*

$$|z|-\frac{|b|}{2-\beta+2\alpha}|z|^2 \leq |f(z)| \leq |z|+\frac{|b|}{2-\beta+2\alpha}|z|^2 \ , \ (|b| \leq 2-\beta+2\alpha) \tag{2.3}$$

and

$$1-\frac{2|b|}{2-\beta+2\alpha}|z| \leq |f'(z)| \leq 1+\frac{2|b|}{2-\beta+2\alpha}|z| \ , \ (|b| \leq \frac{2-\beta+2\alpha}{2}). \tag{2.4}$$

Proof. It is easy to see that, for $f(z) \in G(\alpha, \beta, b)$,

$$\sum_{n=2}^{\infty} a_n \leq \frac{|b|}{2-\beta+2\alpha} \text{ and } \sum_{n=2}^{\infty} n a_n \leq \frac{2|b|}{2-\beta+2\alpha}.$$

$2-\beta+2\alpha \leq \beta+n(1-\beta+\alpha n-\alpha)$ and $\frac{n}{2}(2-\beta+2\alpha) \leq \beta+n(1-\beta+\alpha n-\alpha), (n \geq 2)$, we have

$$|f(z)| \leq |z|+|z|^2\sum_{n=2}^{\infty} a_n \leq |z|+|z|^2\frac{|b|}{2-\beta+2\alpha},$$

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - |z|^2 \frac{|b|}{2-\beta+2\alpha},$$

$$|f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} n a_n \leq 1 + |z| \frac{2|b|}{2-\beta+2\alpha},$$

and

$$|f'(z)| \geq 1 - |z| \sum_{n=2}^{\infty} n a_n \geq 1 - |z| \frac{2|b|}{2-\beta+2\alpha}.$$

□

Theorem 2.3. *Let* $f_i(z) = z - \sum\limits_{n=2}^{\infty} a_{n,i} z^n$ $(a_{n,i} \geq 0,\ i = 1, 2, \cdots, m)$ *be in the class* $G(\alpha, \beta, b)$. *Then the function* $k(z) = \sum_{i=1}^{m} d_i f_i(z)$ $(\sum_{i=1}^{m} d_i = 1)$ *is in the class* $G(\alpha, \beta, b)$.

Proof. By the definition of $k(z)$, we have $k(z) = z - \sum\limits_{n=2}^{\infty} \left[\sum\limits_{i=1}^{m} d_i a_{n,i} z^n \right]$. Thus we have from Theorem 2.1

$$\sum_{n=2}^{\infty} [\beta + n(1-\beta+\alpha n - \alpha)] \left[\sum_{i=1}^{m} d_i a_{n,i} \right] \leq \sum_{i=1}^{m} d_i |b| = |b|,$$

which completes the proof of Theorem 2.3. □

Theorem 2.4. *Let* $\alpha \geq 0$ *and* $|b| \leq |b^*|$. *Then* $G(\alpha, \beta, b) \subset G(\alpha, \beta, b^*)$.

Proof. Assume that $f(z) \in G(\alpha, \beta, b)$. Then $\sum_{n=2}^{\infty} [\beta + n(1-\beta+\alpha n-\alpha)] a_n \leq |b| \leq |b^*|$, which completes the proof of Theorem 2.4. □

Definition 2.1. Let $(f * g)(z)$ denote the Hadamard product of two functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ $(b_n \geq 0)$, that is, $(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$.

Theorem 2.5. *If* $f(z)$ *and* $g(z) \in G(\alpha, \beta, b)$, *then* $(f * g)(z) \in G(\alpha, \beta, b^*)$, *where*

$$|b^*| = \frac{|b|^2}{2-\beta+2\alpha}. \tag{2.5}$$

The result (2.5) is sharp.

Proof. By Theorem 2.1 we have

$$\sum_{n=2}^{\infty} \frac{(\beta + n(1-\beta+\alpha n-\alpha))}{|b|} a_n \leq 1$$

and

$$\sum_{n=2}^{\infty} \frac{(\beta + n(1 - \beta + \alpha n - \alpha))}{|b|} b_n \leq 1. \tag{2.6}$$

we have to find the largest $|b^*|$ such that

$$\sum_{n=2}^{\infty} \frac{(\beta + n(1 - \beta + \alpha n - \alpha))}{|b^*|} \; a_n b_n \leq 1. \tag{2.7}$$

By (2.6) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{n=2}^{\infty} \frac{(\beta + n(1 - \beta + \alpha n - \alpha))}{|b|} \sqrt{a_n b_n} \leq 1. \tag{2.8}$$

Therefore, (2.7) holds true if $\sqrt{a_n b_n} \leq \frac{|b^*|}{|b|}$ for each n. But this is satisfied if

$$\frac{|b|}{(\beta + n(1 - \beta + \alpha n - \alpha))} \leq \frac{|b^*|}{|b|} \text{ or } |b^*| \geq \frac{|b|^2}{(\beta + n(1 - \beta + \alpha n - \alpha))}.$$

But $\psi(n) = \frac{|b^2|}{(\beta + n(1 - \beta + \alpha n - \alpha))}$ is a decreasing function of n. This implies that $|b^*| \geq \psi(2) = \frac{|b|^2}{2 - \beta + 2\alpha}$. □

3. Close-To-Convexity, Starlikeness and Convexity

A function $f(z) \in W$ is said to be close-to-convex of order $\mathcal{E}$ if it satisfies

$$Re\{f'(z)\} > \mathcal{E} \tag{3.1}$$

for some $\mathcal{E}$ $(0 \leq \mathcal{E} < 1)$ and for all $z \in U$. Also a function $f(z) \in W$ is said to be starlike of order $\mathcal{E}$ if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mathcal{E} \tag{3.2}$$

for some $\mathcal{E}$ $(0 \leq \mathcal{E} < 1)$ and for all $z \in U$. Further, a function $f(z) \in W$ is said to be convex of order $\mathcal{E}$, if and only if $zf'(z)$ is starlike of order $\mathcal{E}$, that is if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mathcal{E} \tag{3.3}$$

for some $\mathcal{E}$ $(0 \leq \mathcal{E} < 1)$ and for all $z \in U$.

Theorem 3.1. *If $f(z) \in G(\alpha, \beta, b)$, then $f(z)$ is close-to-convex of order $\mathcal{E}$ in $|z| < r_1(\alpha, \beta, b, \mathcal{E})$, where*

$$r_1(\alpha, \beta, b, \mathcal{E}) = \inf_n \left[\frac{(1 - \mathcal{E})(\beta + n(1 - \beta + \alpha n - \alpha))}{|b|}\right]^{\frac{1}{n-1}}.$$

Proof. It is sufficient to show that

$$|f'(z)-1| < \sum_{n=2}^{\infty} na_n|z|^{n-1} \leq 1-\mathcal{E} \tag{3.4}$$

and

$$\sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]a_n \leq |b| \tag{3.5}$$

observe that (3.4) is true if

$$\frac{n|z|^{n-1}}{1-\mathcal{E}} \leq \frac{(\beta+n(1-\beta+\alpha n-\alpha))}{|b|}. \tag{3.6}$$

Solving (3.6) for $|z|$, we obtain

$$|z| \leq \left[\frac{(1-\mathcal{E})(\beta+n(1-\beta+\alpha n-\alpha))}{|b|}\right]^{\frac{1}{n-1}}, \ n=2,3,\cdots .$$

□

Theorem 3.2. *If $f(z) \in G(\alpha,\beta,b)$, then $f(z)$ is starlike of order $\mathcal{E}$ in $|z| < r_2(\alpha,\beta,b,\mathcal{E})$, where*

$$r_2(\alpha,\beta,b,\mathcal{E}) = \inf_n \left[\frac{(1-\mathcal{E})(\beta+n(1-\beta+\alpha n-\alpha))}{(n-\mathcal{E})|b|}\right]^{\frac{1}{n-1}}.$$

Proof. We must show that $\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\mathcal{E}$ for $|z| < r_2(\alpha,\beta,b,\mathcal{E})$. Since

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}, \tag{3.7}$$

if $\frac{(n-\mathcal{E})|z|^{n-1}}{1-\mathcal{E}} \leq \frac{(\beta+n(1-\beta+\alpha n-\alpha))}{|b|}$, $f(z)$ is starlike of order $\mathcal{E}$. □

Corollary 3.3. *If $f(z) \in G(\alpha,\beta,b)$, then $f(z)$ is convex of order $\mathcal{E}$ in $|z| < r_3(\alpha,\beta,b,\mathcal{E})$, where*

$$r_3(\alpha,\beta,b,\mathcal{E}) = \inf_n \left[\frac{(1-\mathcal{E})(\beta+n(1-\beta+\alpha n-\alpha))}{n(n-\mathcal{E})|b|}\right]^{\frac{1}{n-1}}.$$

4. An Application of Fractional Calculus

The following definition for the fractional calculus was given by Owa[2]. For other definitions see, for example, the references cited by Srivastava et al. ([4], [5], [7]).

Definition 4.1. The fractional integral of order δ is defined by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\gamma) d\gamma}{(z-\gamma)^{1-\delta}}$$

where $\delta > 0, f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\gamma)^{\delta-1}$ is removed by required $\log(z-\gamma)$ to be real when $z-\gamma > 0$.

Definition 4.2. The fractional derivative of order δ is defined by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\gamma) d\gamma}{(z-\gamma)^{\delta}}$$

where $0 \leq \delta < 1, f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\gamma)^{-\delta}$ is removed as in Definition 4.1 above.

Definition 4.3. Under the conditions of Definition 4.2, the fractional derivative of order $n+\delta$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z),$$

where $0 \leq \delta < 1$ and $n = 0, 1, \cdots$.

Theorem 4.1. *Let the function $f(z)$ be in the class $G(\alpha, \beta, b)$. Then*

$$|D_z^{-\delta} f(z)| \leq \frac{1}{\Gamma(2+\delta)} |z|^{1+\delta} \left[1 + \frac{2|b|}{(2-\beta+2\alpha)(2+\delta)} |z|\right] \tag{4.1}$$

and

$$|D_z^{-\delta} f(z)| \geq \frac{1}{\Gamma(2+\delta)} |z|^{1+\delta} \left[1 - \frac{2|b|}{(2-\beta+2\alpha)(2+\delta)} |z|\right], \tag{4.2}$$

$\left(|b| \leq \frac{(2-\beta+2\alpha)(2+\delta)}{2\Gamma(2+\delta)}\right)$.

The equalities in (4.1) and (4.2) are attained for the function

$$f(z) = z - \frac{|b|}{2-\beta+2\alpha} z^2 \tag{4.3}$$

Proof. Using Theorem 2.1 we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{|b|}{2-\beta+2\alpha}. \tag{4.4}$$

From Definition 4.1 we get

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{1+\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{n+\delta}$$

and

$$\begin{aligned} \Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} a_n z^n \qquad (4.5) \\ &= z - \sum_{n=2}^{\infty} \Psi(n) a_n z^n \end{aligned}$$

where $\Psi(n) = \dfrac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}$.

We know that $\Psi(n)$ is a decreasing function of n and $0 < \Psi(n) \leq \Psi(2) = \frac{2}{2+\delta}$. Using (4.4) and (4.5) we have

$$\begin{aligned} |\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)| &\leq |z| + \Psi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{2|b|}{(2-\beta+2\alpha)(2+\delta)} |z|^2, \end{aligned}$$

which gives (4.1); we also have

$$\begin{aligned} |\Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z)| &\geq |z| - \Psi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{2|b|}{(2-\beta+2\alpha)(2+\delta)} |z|^2, \end{aligned}$$

which gives (4.2). □

Theorem 4.2. *Let the function $f(z)$ be in the class $G(\alpha,\beta,b)$. Then*

$$|D_z^{\delta} f(z)| \leq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[1 + \frac{2|b|}{(2-\beta+2\alpha)(2-\delta)} |z|\right] \tag{4.6}$$

and

$$|D_z^{\delta} f(z)| \geq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[1 - \frac{2|b|}{(2-\beta+2\alpha)(2-\delta)} |z|\right], \tag{4.7}$$

$\left(|b| \leq \frac{(2-\beta+2\alpha)(2-\delta)}{2\Gamma(2-\delta)}\right)$.

The equalities in (4.6) and (4.7) are attained for the function $f(z)$ given by (4.3).

Proof. Using Theorem 2.1 we have

$$\sum_{n=2}^{\infty} na_n \leq \frac{2|b|}{2-\beta+2\alpha}. \tag{4.8}$$

By Definition 4.2 we get

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta}$$

and

$$\Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n$$

$$= z - \sum_{n=2}^{\infty} \frac{\Gamma(n)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} n a_n z^n = z - \sum_{n=2}^{\infty} n\Phi(n) a_n z^n \tag{4.9}$$

since

$$\Phi(n) = \frac{\Gamma(n)\Gamma 2-\delta)}{\Gamma(n+1-\delta)}$$

is a decreasing function of n and

$$0 < \Phi(n) \leq \Phi(2) = \frac{1}{2-\delta},$$

using (4.8) and (4.9), we have

$$\begin{aligned} |\Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z)| &\leq |z| + \Phi(2)|z|^2 \sum_{n=2}^{\infty} na_n \\ &\leq |z| + \frac{2|b|}{(2-\beta+2\alpha)(2-\delta)} |z|^2, \end{aligned}$$

which gives (4.6); and

$$\begin{aligned} |\Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z)| &\geq |z| - \Phi(z)|z|^2 \sum_{n=2}^{\infty} na_n \\ &\geq |z| - \frac{2|b|}{(2-\beta+2\alpha)(2-\delta)} |z|^2, \end{aligned}$$

which gives (4.7). □

Corollary 4.3. *By letting $\delta = 0$ in Theorem 4.1 and letting $\delta = 1$ in Theorem 4.2, we have Theorem 2.2.*

Corollary 4.4. *Under the hypotheses of Theorem 4.1 and 4.2,* $D_z^{-\delta} f(z)$ *and* $D_z^{\delta} f(z)$ *are included in the disk with center at the origin and radii*

$$\frac{1}{\Gamma(2+\delta)}\left[1+\frac{2|b|}{(2-\beta+2\alpha)(2+\delta)}\right] \text{ and } \frac{1}{\Gamma(2-\delta)}\left[1+\frac{2|b|}{(2-\beta+2\alpha)(2-\delta)}\right],$$

respectively.

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