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On a Certain Subclass of Univalent Functions Defined by Differential Subordination Property

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Abstract

The object of the present paper is to investigate and study certain subclass of univalent functions defined by differential subordination by using the linear operator $\mathcal{L}_{\lambda,\iota,m}^{t,\alpha_1}$. *Coefficient bounds, some properties of neighborhoods, convolution properties, Integral mean inequalities for the fractional integral for this certain subclass are given.*

Keywords: *Univalent Function, Differential Subordination, δ- neighborhood, Convolution, Hypergeometric Function, Linear Operator, Integral Mean, Fractional Integral.*

1 Introduction

Let G be the class of all functions of the form:-

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (n \in N), \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.
Let $\mathcal{A}$ denote the subclass of G containing of functions of the form:-

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, \quad n \in N). \tag{1.2}$$

The Hadamard product (or convolution) of two power series

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad and \quad g(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{1.3}$$

in $\mathcal{A}$ is defined (as usual)by

$$(f * g)(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.4}$$

For positive real values of $\alpha_1, \dots, \alpha_\iota$ *and* $\beta_1, \dots, \beta_m (\beta_j \neq 0, -1, \dots, j = 1,2, \dots, m)$,
the generalized hypergeometric function ${}_\iota F_m(z)$ is defined by

$${}_\iota F_m(z) \equiv {}_\iota F_m(\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_\iota)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \tag{1.5}$$

$(\iota \leq m + 1; \iota, m \epsilon N_0 = N \cup \{0\}; z \in U)$,

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2) \dots (a+n-1), & a \in N. \end{cases} \tag{1.6}$$

The notation ${}_\iota F_m$ is quite useful for representing many well- known functions such as the exponential, the Binomial, the Bessel and Laguerre polynomial. Let

$$H[\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m]: \mathcal{A} \longrightarrow \mathcal{A}$$

be a linear operator defined by

$$H[\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m]f(z) = z \ {}_\iota F_m(\alpha_1, \alpha_2, \dots, \alpha_\iota; \beta_1, \beta_2, \dots, \beta_m; z) * f(z)$$

$$= z - \sum_{n=2}^{\infty} w_n(\alpha_1; \iota; m) a_n z^n, \qquad (1.7)$$

Where,

$$w_n(\alpha_1; \iota; m) = \frac{(\alpha_1)_{n-1\dots}(\alpha_\iota)_{n-1}}{(\beta_1)_{n-1\dots}(\beta_m)_{n-1}} \frac{1}{(n-1)!} . \qquad (1.8)$$

For notational simplicity, we use shorter notation $H_m^\iota[\alpha_1]$ for $H[\alpha_1, \dots, \alpha_\iota; \beta_1, \dots, \beta_m]$.

In the sequel. It follows from (1.7) that

$$H_0^1[1]f(z) = f(z), \ \ H_0^1[2]f(z) = zf'(z).$$

The linear operator $H_m^\iota[\alpha_1]$ is called Dziok–Srivastava operator (see[3]) introduced by Dziok and Srivastava which was subsequently extended by Dziok and Raina [2] by using the generalized hypergeometric function, recently Srivastava *et. al.* ([10]) defined the linear operator $\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1}$ as follows:-

$$\mathcal{L}_{\lambda,\alpha_1}^{0} f(z) = f(z)$$

$$\mathcal{L}_{\lambda,\iota,m}^{1,\alpha_1} f(z) = (1-\lambda) H_m^\iota[\alpha_1] f(z) + \lambda (H_m^\iota[\alpha_1] f(z))'$$

$$= \mathcal{L}_{\lambda,\iota,m}^{\alpha_1} f(z), \qquad (\lambda \geq 0), \qquad (1.9)$$

$$\mathcal{L}_{\lambda,\iota,m}^{2,\alpha_1} f(z) = \mathcal{L}_{\lambda,\iota,m}^{\alpha_1} \left(\mathcal{L}_{\lambda,\iota,m}^{1,\alpha_1} f(z) \right) \qquad (1.10)$$

and in general ,

$$\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1} f(z) = \mathcal{L}_{\lambda,\iota,m}^{\alpha_1} \left(\mathcal{L}_{\lambda,\iota,m}^{\tau-1,\alpha_1} f(z) \right), \ \ (\iota \leq m+1; \iota, m \in N_0 = N \cup \{0\}; z \in U) \qquad (1.11)$$

If the function $f(z)$ is given by (1.2), then we see form (1.7), (1.8), (1.9) and (1.11) that

$$\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1} f(z) = z - \sum_{n=2}^{\infty} w_n^{\tau}(\alpha_1;\lambda;\iota;m) a_n z^n, \tag{1.12}$$

where,

$$w_n^{\tau}(\alpha_1;\lambda;\iota;m) = \left(\frac{(\alpha_1)_{n-1\dots}(\alpha_\iota)_{n-1}}{(\beta_1)_{n-1\dots}(\beta_m)_{n-1}} \frac{[1+\lambda(n-1)]}{(n-1)!}\right)^{\tau},$$

$$(n \in N\backslash\{1\}, \tau \epsilon N_0). \tag{1.13}$$

Unless otherwise stated. We note that when $\tau = 1$ *and* $\lambda = 0$ the linear operator $\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1}$ would reduce to the familier Dziok – Srivastava linear operator given by (see [3]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer[1], Owa[7] and Ruscheweyh[8].

For two analytic functions $f, g \in \mathcal{A}$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with

$$w(0) = 0 \ \textit{and} \ |w(z)| < 1 \text{ for all z} \in \text{U, such that } f(z) = g\big(w(z)\big), z \in \text{U}.$$

Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence (see [6]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \ \textit{and} \ f(U) \subset g(U).$$

Definition 1: *For any function* $f \in \mathcal{A}$ *and* $\delta \geq 0$, *the* $\delta-$ *neighborhood of* f *is defined as,*

$$N_\delta(f) = \left\{ g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}: \ \sum_{n=2}^{\infty} n \left| a_n - b_n \right| \leq \delta \right\}. \tag{1.14}$$

In particular, for the function $e(z) = z$, we see that,

$$N_\delta(e) = \left\{ g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}: \ \sum_{n=2}^{\infty} n \left| b_n \right| \leq \delta \right\}. \tag{1.15}$$

The concept of neighborhoods was first introduced by Goodman [4] and then generalized by Ruscheweyh [9].

Definition 2: *For fixed parameters A and B, with* $-1 \leq B < A \leq 1$, *we say that* $f \in \mathcal{A}$ *is in the class* $W(\tau, \theta, \alpha_1, \lambda, \iota, m, A, B)$ *if it satisfies the following subordination condition:*

$$\frac{\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1} f(z)}{\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1} f(z)} \prec \frac{1+\mathrm{A}z}{1+\mathrm{B}z} \quad . \tag{1.16}$$

In view of the definition of subordination, (1.16) is equivalent to the following condition:

$$\left| \frac{\frac{\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1} f(z)}{\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1} f(z)} - 1}{\mathrm{B}\frac{\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1} f(z)}{\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1} f(z)} - \mathrm{A}} \right| < 1, (z \in U).$$

For convenience, we write
$W(\tau,\theta,\alpha_1,\lambda,\iota,m,1-2\eta,-1) = W(\tau,\theta,\alpha_1,\lambda,\iota,m,\eta)$,

where $W(\tau,\theta,\alpha_1,\lambda,\iota,m,\eta)$ denotes the class of functions in $\mathcal{A}$ satisfying the inequality:

$$Re\left\{\frac{\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1} f(z)}{\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1} f(z)}\right\} > \eta, \quad (0 \le \eta < 1; \quad z \in U).$$

2 Neighborhoods for the Class $W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$

Theorem 2.1: *A function $f \in \mathcal{A}$ belongs to the class $W(\tau,\theta,\alpha_1,\lambda,\iota,m,A,B)$ if and only if*

$$\sum_{n=2}^{\infty} w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left\{(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m) - (1-\mathrm{A})\right\}a_n \le \mathrm{A}-\mathrm{B} \tag{2.1}$$

for $\tau,\theta,\iota,m \in N_0, \iota \le m+1, \lambda \ge 0$ *and* $-1 \le B < A \le 1$.

Proof: Let $f \in W(\tau,\theta,\alpha_1,\lambda,t,m,\mathrm{A},\mathrm{B})$. Then,

$$\frac{\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1} f(z)}{\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1} f(z)} \prec \frac{1+\mathrm{A}z}{1+\mathrm{B}z} \; z \in U . \tag{2.2}$$

Therefore, there exists an analytic function w such that

$$w(z)=\frac{\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1}f(z)-\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1}f(z)}{\mathrm{B}\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1}f(z)-\mathrm{A}\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1}f(z)}\,. \tag{2.3}$$

Hence,

$$|w(z)|=\left|\frac{\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1}f(z)-\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1}f(z)}{\mathrm{B}\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1}f(z)-\mathrm{A}\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1}f(z)}\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\{w_n^{\theta}(\alpha_1;\lambda;\iota;m)-1\}a_nz^n}{(\mathrm{A}-\mathrm{B})z+\sum_{n=2}^{\infty}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\{\mathrm{B}w_n^{\theta}(\alpha_1;\lambda;\iota;m)-\mathrm{A}\}a_nz^n}\right|<1.$$

Thus,

$$Re\left\{\frac{\sum_{n=2}^{\infty}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\{w_n^{\theta}(\alpha_1;\lambda;\iota;m)-1\}a_nz^n}{(\mathrm{A}-\mathrm{B})z+\sum_{n=2}^{\infty}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\{\mathrm{B}w_n^{\theta}(\alpha_1;\lambda;\iota;m)-\mathrm{A}\}a_nz^n}\right\}<1. \tag{2.4}$$

Taking $|z|=r$, for sufficiently small r with $0<r<1$, the denominator of (2.4) is positive and so it is positive for all r with $0<r<1$, since $w(z)$ is analytic for $|z|<1$. Then, the inequality (2.4) yields

$$\sum_{n=2}^{\infty}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\{w_n^{\theta}(\alpha_1;\lambda;\iota;m)-1\}a_nr^n$$

$$<(\mathrm{A}-\mathrm{B})r+\sum_{n=2}^{\infty}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\{\mathrm{B}w_n^{\theta}(\alpha_1;\lambda;\iota;m)-\mathrm{A}\}a_nr^n.$$

Equivalently,

$$\sum_{n=2}^{\infty}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\{(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\}a_nr^n\le(\mathrm{A}-\mathrm{B})r,$$

and (2.1) follows upon letting $r\to1$.

Conversely, for $|z|=r$, $0<r<1$, we have $r^n<r$. That is,

$$\sum_{n=2}^{\infty}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\{(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\}a_nr^n$$

$$\leq \sum_{n=2}^{\infty} w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left\{(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right\}a_n r \leq (\mathrm{A}-\mathrm{B})r.$$

From (2.1), we have

$$\left|\sum_{n=2}^{\infty} w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left\{w_n^{\theta}(\alpha_1;\lambda;\iota;m)-1\right\}a_n z^n\right|$$

$$\leq \sum_{n=2}^{\infty} w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left\{w_n^{\theta}(\alpha_1;\lambda;\iota;m)-1\right\}a_n r^n$$

$$< (\mathrm{A}-\mathrm{B})r + \sum_{n=2}^{\infty}\left\{\mathrm{B}w_n^{\theta}(\alpha_1;\lambda;\iota;m)-\mathrm{A}\right\}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\, a_n r^n$$

$$< \left|(\mathrm{A}-\mathrm{B})z + \sum_{n=2}^{\infty}\left\{\mathrm{B}w_n^{\theta}(\alpha_1;\lambda;\iota;m)-\mathrm{A}\right\}w_n^{\tau}(\alpha_1;\lambda;\iota;m)\, a_n z^n\right|.$$

This proves that

$$\frac{\mathcal{L}_{\lambda,\iota,m}^{\tau+\theta,\alpha_1}f(z)}{\mathcal{L}_{\lambda,\iota,m}^{\tau,\alpha_1}f(z)} \prec \frac{1+\mathrm{A}z}{1+\mathrm{B}z}, \qquad z \in U$$

and hence $f \in W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$.

Theorem 2.2 *If*

$$\delta = \frac{(\mathrm{A}-\mathrm{B})}{\left(\frac{(\alpha_1)_1\cdots(\alpha_\iota)_1}{(\beta_1)_1\cdots(\beta_m)_1}(1+\lambda)\right)^{\tau-1}\left[(1-\mathrm{B})\left(\frac{(\alpha_1)_1\cdots(\alpha_\iota)_1}{(\beta_1)_1\cdots(\beta_m)_1}(1+\lambda)\right)^{\theta}-(1-\mathrm{A})\right]}, \tag{2.5}$$

then $W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B}) \subset N_\delta(e)$.

Proof: It follows from (2.1), that if $f \in W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$, then

$$w_2^{\tau-1}(\alpha_1;\lambda;\iota;m)\left\{(1-\mathrm{B})w_2^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right\}\sum_{n=2}^{\infty} n\, a_n \leq (\mathrm{A}-\mathrm{B}),$$

Hence

$$\left(\frac{(\alpha_1)_1\cdots(\alpha_\iota)_1}{(\beta_1)_1\cdots(\beta_m)_1}(1+\lambda)\right)^{\tau-1}\left\{(1-\mathrm{B})\left(\frac{(\alpha_1)_1\cdots(\alpha_\iota)_1}{(\beta_1)_1\cdots(\beta_m)_1}(1+\lambda)\right)^{\theta} - (1-\mathrm{A})\right\}\sum_{n=2}^{\infty} n\, a_n \leq (\mathrm{A}-\mathrm{B}). \tag{2.6}$$

Which implies,

$$\sum_{n=2}^{\infty} n\, a_n$$
$$\leq \frac{(\mathrm{A}-\mathrm{B})}{\left(\frac{(\alpha_1)_1 \dots (\alpha_\iota)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^{\tau-1}\left[(1-\mathrm{B})\left(\frac{(\alpha_1)_1 \dots (\alpha_\iota)_1}{(\beta_1)_1 \dots (\beta_m)_1}(1+\lambda)\right)^{\theta}-(1-\mathrm{A})\right]}$$
$$= \delta. \tag{2.7}$$

Using (1.15), we get the result.

Definition (2.1): *The function* g *defined by*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n$$

is said to be a member of the class $W_y(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$ if there exists a function $f \in W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$ such that

$$\left|\frac{g(z)}{f(z)} - 1\right| \leq 1 - y, \ (z \in U, 0 \leq y < 1). \tag{2.8}$$

Theorem (2.3): If $f \in W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$ and

$$y = 1 - \frac{\delta w_2^{\tau}(\alpha_1;\lambda;\iota;m)\{(1-\mathrm{B})w_2^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\}}{2(w_2^{\tau}(\alpha_1;\lambda;\iota;m)\{(1-\mathrm{B})w_2^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\}-(\mathrm{A}-\mathrm{B}))}, \tag{2.9}$$

then $N_\delta(f) \subset W_y(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$.

Proof: Let $g \in N_\delta(f)$. Then we have from (1.14) that

$$\sum_{n=2}^{\infty} n|\, a_n - b_n\,| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |\, a_n - b_n\,| \leq \frac{\delta}{2}\ .$$

Also since $f \in W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$, we have from (2.1)

$$\sum_{n=2}^{\infty} a_n \leq \frac{(\mathrm{A}-\mathrm{B})}{w_2^{\tau}(\alpha_1;\lambda;\iota;m)[(1-\mathrm{B})w_2^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})]}\ ,$$

where

$$w_2^{\tau}(\alpha_1;\lambda;\iota;m)=\left(\frac{(\alpha_1)_1\dots(\alpha_\iota)_1}{(\beta_1)_1\dots(\beta_m)_1}(1+\lambda)\right)^{\tau},$$

$$w_2^{\theta}(\alpha_1;\lambda;\iota;m)=\left(\frac{(\alpha_1)_1\dots(\alpha_\iota)_1}{(\beta_1)_1\dots(\beta_m)_1}(1+\lambda)\right)^{\theta}.$$

So that

$$\left|\frac{g(z)}{f(z)}-1\right|=\left|\frac{\sum_{n=2}^{\infty}(a_n-b_n)z^n}{z-\sum_{n=2}^{\infty}a_nz^n}\right|<\frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}a_n}$$

$$\le\frac{\delta}{2}\cdot\frac{w_2^{\tau}(\alpha_1;\lambda;\iota;m)[(1-\mathrm{B})w_2^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})]}{w_2^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-\mathrm{B})w_2^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\big]-(\mathrm{A}-\mathrm{B})}$$

=1-y .

Thus by Definition (2.1), $g\in W_y(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$ for y given by (2.9). This completes the proof.

3 Convolution Properties:

Theorem 3.1: Let the functions f_j $(j=1,2)$ defined by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}\,z^n,\quad \left(a_{n,j}\ge 0, j=1,2\right)\quad, \tag{3.1}$$

be in the class $W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$.

Then $f_1*f_2\in W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\sigma)$, where

$$\sigma\le\frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\big]^2\mathrm{A}-(\mathrm{A}-\mathrm{B})^2(w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A}))}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})]^2-w_n^{\theta}(\alpha_1;\lambda;\iota;m)(\mathrm{A}-\mathrm{B})^2}$$

Proof: We must find the largest σ such that

$$\sum_{n=2}^{\infty}\frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-\sigma)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\big]}{\mathrm{A}-\sigma}\;a_{n,1}a_{n,2}\le 1.$$

Since $f_j\in W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\mathrm{B})$ $(j=1,2)$, then

$$\sum_{n=2}^{\infty} \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}{\mathrm{A}-\mathrm{B}}\ a_{n,j} \leq 1,$$

$$(j=1,2). \tag{3.2}$$

By Cauchy-Schwarz inequality, we get

$$\sum_{n=2}^{\infty} \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}{\mathrm{A}-\mathrm{B}} \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{3.3}$$

We want only to show that

$$\frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\sigma)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}{\mathrm{A}-\sigma} a_{n,1}a_{n,2}$$

$$\leq \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}{\mathrm{A}-\mathrm{B}} \sqrt{a_{n,1}a_{n,2}}\ .$$

This equivalently to

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(\mathrm{A}-\sigma)\left[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}{(\mathrm{A}-\mathrm{B})\left[(1-\sigma)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}.$$

From (3.3), we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{\mathrm{A}-\mathrm{B}}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}.$$

Thus, it is sufficient to show that

$$\frac{\mathrm{A}-\mathrm{B}}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}$$

$$\leq \frac{(\mathrm{A}-\sigma)\left[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}{(\mathrm{A}-\mathrm{B})\left[(1-\sigma)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]},$$

Which implies to

$$\sigma \leq \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]^2\mathrm{A}-(\mathrm{A}-\mathrm{B})^2\left(w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right)}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})]^2-w_n^{\theta}(\alpha_1;\lambda;\iota;m)(\mathrm{A}-\mathrm{B})^2}.$$

Theorem (3.2): *Let the functions* f_j $(j = 1,2)$ *defined by (3.1) be in the class* $\mathrm{W}(\tau, \theta, \alpha_1, \lambda, \iota, \mathrm{m}, \mathrm{A}, \mathrm{B})$. *Then the function* k *defined by*

$$k(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \tag{3.4}$$

belong to the class $W(\tau, \theta, \alpha_1, \lambda, \iota, m, \mathrm{A}, \varepsilon)$, where

$$\varepsilon \leq \frac{\mathrm{A}(w_n^\tau(\alpha_1; \lambda; \iota; m))^2 \left[(1-\mathrm{B}) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1-\mathrm{A})\right]^2 - 2(\mathrm{A}-\mathrm{B})^2 w_n^{\tau+\theta}(\alpha_1; \lambda; \iota; m) + 2(\mathrm{A}-\mathrm{B})^2 (1-\mathrm{A}) w_n^\tau(\alpha_1; \lambda; \iota; m)}{(w_n^\tau(\alpha_1; \lambda; \iota; m))^2 [(1-\mathrm{B}) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1-\mathrm{A})]^2 - 2(\mathrm{A}-\mathrm{B})^2 w_n^{\tau+\theta}(\alpha_1; \lambda; \iota; m)}$$

Proof: We must find the largest ε such that

$$\sum_{n=2}^{\infty} \frac{w_n^\tau(\alpha_1; \lambda; \iota; m)\left[(1-\varepsilon) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1-\mathrm{A})\right]}{\mathrm{A} - \varepsilon} (a_{n,1}^2 + a_{n,2}^2) \leq 1.$$

Since $f_j \in W(\tau, \theta, \alpha_1, \lambda, \iota, m, \mathrm{A}, \mathrm{B})$ $(j = 1,2)$, we get

$$\sum_{n=2}^{\infty} \left(\frac{w_n^\tau(\alpha_1; \lambda; \iota; m)\left[(1-\mathrm{B}) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1-\mathrm{A})\right]}{\mathrm{A} - \mathrm{B}} \right)^2 a_{n,1}^2$$
$$\leq \left(\sum_{n=2}^{\infty} \frac{w_n^\tau(\alpha_1; \lambda; \iota; m)\left[(1-\mathrm{B}) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1-\mathrm{A})\right]}{\mathrm{A} - \mathrm{B}} a_{n,1} \right)^2$$
$$\leq 1, \tag{3.5}$$

and

$$\sum_{n=2}^{\infty} \left(\frac{w_n^\tau(\alpha_1; \lambda; \iota; m)\left[(1-\mathrm{B}) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1-\mathrm{A})\right]}{\mathrm{A} - \mathrm{B}} \right)^2 a_{n,2}^2$$
$$\leq \left(\sum_{n=2}^{\infty} \frac{w_n^\tau(\alpha_1; \lambda; \iota; m)\left[(1-\mathrm{B}) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1-\mathrm{A})\right]}{\mathrm{A} - \mathrm{B}} a_{n,2} \right)^2$$
$$\leq 1. \tag{3.6}$$

Combining the inequalities (3.5) and (3.6), gives

$$\sum_{n=2}^{\infty} \frac{1}{2} \left(\frac{w_n^\tau(\alpha_1; \lambda; \iota; m)\left[(1-\mathrm{B}) w_n^\theta(\alpha_1; \lambda; \iota; m) - (1-\mathrm{A})\right]}{\mathrm{A} - \mathrm{B}} \right)^2 \left(a_{n,1}^2 + a_{n,2}^2\right)$$
$$\leq 1. \tag{3.7}$$

But, k $\in W(\tau,\theta,\alpha_1,\lambda,\iota,m,\mathrm{A},\varepsilon)$, if and only if

$$\sum_{n=2}^{\infty}\frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\varepsilon)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\right]}{\mathrm{A}-\varepsilon}(a_{n,1}^2+a_{n,2}^2) \le 1\,. \tag{3.8}$$

The inequality (3.8) will be satisfied if

$$\frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-\varepsilon)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\right]}{A-\varepsilon} \le \frac{(w_n^{\tau}(\alpha_1;\lambda;\iota;m))^2\left[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\right]^2}{2(A-B)^2},$$

(n=2,3,…)

so that

$$\boldsymbol{\varepsilon} \le \frac{\mathrm{A}(w_n^{\tau}(\alpha_1;\lambda;\iota;m))^2[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})]^2-2(\mathrm{A}-\mathrm{B})^2w_n^{\tau+\theta}(\alpha_1;\lambda;\iota;m)+2(\mathrm{A}-\mathrm{B})^2(1-\mathrm{A})w_n^{\tau}(\alpha_1;\lambda;\iota;m)}{(w_n^{\tau}(\alpha_1;\lambda;\iota;m))^2[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})]^2-2(\mathrm{A}-\mathrm{B})^2w_n^{\tau+\theta}(\alpha_1;\lambda;\iota;m)}$$

4 Integral Mean Inequalities for the Fractional Integral:

Definition (4.1) [6]: *The fractional integral of order s $(s>0)$ is defined for a function f by*

$$D_z^{-s}f(z)=\frac{1}{\Gamma(s)}\int_0^z\frac{f(t)}{(z-t)^{1-s}}\,dt\,, \tag{4.1}$$

where the function f is an analytic in a simply – connected region of the complex z-plane containing the origin, and multiplicity of $(z-t)^{s-1}$ is removed by requiring $log\,(z-t)$ to be real, when $(z-t)>0$.

In 1925, Littlewood [5] proved the following subordination theorem:

Theorem 4.1 (Littlewood [5]): If f *and* g are analytic in U with $f\prec g$, then for

$$\mu>0 \text{ and } z=re^{i\theta}\ (0<r<1)$$

$$\int_0^{2\pi}|f(z)|^{\mu}\,d\theta \le \int_0^{2\pi}|g(z)|^{\mu}\,d\theta.$$

Theorem 4.2: Let $f\in W(\tau,\theta,\alpha_1,\lambda,\iota,m,A,B)$ and suppose that f_n is defined by

$$f_n(z) = z - \frac{A-B}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\right]} z^n, \quad (n \geq 2). \tag{4.2}$$

Also let

$$\sum_{i=2}^{\infty}(i-\eta)_{\eta+1}a_i \leq \frac{(A-B)\Gamma(n+1)\Gamma(s+\eta+3)}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)]\Gamma(n+s+\eta+1)\Gamma(2-\eta)}, \tag{4.3}$$

for $0 \leq \eta \leq i$, $s > 0$, where $(i-\eta)_{\eta+1}$ denote the pochhammer symbol defined by $(i-\eta)_{\eta+1} = (i-\eta)(i-\eta+1)\ldots i$.

If there exists an analytic function q defined by

$$(q(z))^{n-1} = \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\right]\Gamma(n+s+\eta+1)}{(A-B)\Gamma(n+1)} \sum_{i=p+1}^{\infty} (i-\eta)_{\eta+1} H(i)a_i z^{i-1}, \tag{4.5}$$

where $i \geq \eta$ and

$$H(i) = \frac{\Gamma(i-\eta)}{\Gamma(i+s+\eta+1)}, \qquad (s > 0 \ , i \geq 2)\,, \tag{4.6}$$

then, for $z = re^{i\gamma}$ and $0 < r < 1$

$$\int_0^{2\pi} \left|D_z^{-s-\eta} f(z)\right|^{\mu} d\gamma \leq \int_0^{2\pi} \left|D_z^{-s-\eta} f_n(z)\right|^{\mu} d\gamma, \quad (s > 0, \mu > 0). \tag{4.7}$$

Proof: Let

$$f(z) = z - \sum_{i=2}^{\infty} a_i z^i \quad .$$

For $\eta \geq 0$ and Definition(4.1), we get

$$D_z^{-s-\eta} f(z) = \frac{\Gamma(2)z^{s+\eta+1}}{\Gamma(s+\eta+2)}\left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(i+1)\Gamma(s+\eta+2)}{\Gamma(2)\Gamma(i+s+\eta+1)} a_i z^{i-1}\right)$$

$$= \frac{\Gamma(2) z^{s+\eta+1}}{\Gamma(s+\eta+2)} \left(1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\eta+2)}{\Gamma(2)} (i-\eta)_{\eta+1} H(i) a_i z^{i-1} \right),$$

where

$$H(i) = \frac{\Gamma(i-\eta)}{\Gamma(i+s+\eta+1)}, \qquad (s > 0\ , i \geq 2).$$

Since H is a decreasing function of i, we have

$$0 < H(i) \leq H(2) = \frac{\Gamma(2-\eta)}{\Gamma(s+\eta+3)}.$$

Similarly, from (4.2) and Definition4.1, we get

$$D_z^{-s-\eta} f_n(z) = \frac{\Gamma(2) z^{s+\eta+1}}{\Gamma(s+\eta+2)} \left(1 - \frac{(A-B)\Gamma(n+1)\Gamma(s+\eta+2)}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-B) w_n^{\theta}(\alpha_1;\lambda;\iota;m) - (1-A)\right]\Gamma(n+s+\eta+1)} z^{n-1} \right).$$

For $\mu > 0$ and $z = re^{i\gamma}$ $(0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\eta+2)}{\Gamma(2)} (i-\eta)_{\eta+1} H(i) a_i z^{i-1} \right|^{\mu} d\gamma$$

$$\leq \int_0^{2\pi} \left| 1 - \frac{(A-B)\Gamma(n+1)\Gamma(s+\eta+2)}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)[(1-B) w_n^{\theta}(\alpha_1;\lambda;\iota;m) - (1-A)]\Gamma(2)\Gamma(n+s+\eta+1)} z^{n-1} \right|^{\mu} d\gamma.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\eta+2)}{\Gamma(2)} (i-\eta)_{\eta+1} H(i) a_i z^{i-1}$$

$$\prec 1 - \frac{(A-B)\Gamma(n+1)\Gamma(s+\eta+2)}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\left[(1-B) w_n^{\theta}(\alpha_1;\lambda;\iota;m) - (1-A)\right]\Gamma(2)\Gamma(n+s+\eta+1)} z^{n-1}.$$

By setting

$$1 - \sum_{i=2}^{\infty} \frac{\Gamma(s+\eta+2)}{\Gamma(2)} (i-\eta)_{\eta+1} H(i) a_i z^{i-1}$$

$$= 1 - \frac{(A-B)\Gamma(n+1)\Gamma(s+\eta+2)}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\big]\Gamma(2)\Gamma(n+s+\eta+1)}(q(z))^{n-1},$$

we find that

$$(q(z))^{n-1} = \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\big]\Gamma(n+s+\eta+1)}{(A-B)\Gamma(n+1)}\sum_{i=2}^{\infty}(i-\eta)_{\eta+1}H(i)a_i z^{i-1},$$

which readily yields $w(0)=0$. For such a function q, we obtain

$$|(q(z))|^{n-1} \le \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\big]\Gamma(n+s+\eta+1)}{(A-B)\Gamma(n+1)}\sum_{i=2}^{\infty}(i-\eta)_{\eta+1}H(i)a_i|z|^{i-1}$$

$$\le \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\big]\Gamma(n+s+\eta+1)}{(A-B)\Gamma(n+1)}H(2)|z|\sum_{i=2}^{\infty}(i-\eta)_{\eta+1}H(i)a_i$$

$$= |z|\frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\big]\Gamma(n+s+\eta+1)\Gamma(2-\eta)}{(A-B)\Gamma(s+\eta+3)\Gamma(n+1)}\sum_{i=2}^{\infty}(i-\eta)_{\eta+1}a_i \le |z| < 1.$$

This completes the proof of the theorem .
By taking $\eta = 0$ in the Theorem4.2, we have the following corollary :

Corollary 4.1: *Let $f \in W(\tau,\theta,\alpha_1,\lambda,\iota,m,A,B)$ and suppose that f_n is defined by (4.2). Also let*

$$\sum_{i=2}^{\infty} i a_i \le \frac{(A-B)\Gamma(n+1)\Gamma(s+3)}{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-B)w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-A)\big]\Gamma(n+s+1)\Gamma(2)}, n \ge 2.$$

If there exists an analytic function q defined by

$$(q(z))^{n-1} = \frac{w_n^{\tau}(\alpha_1;\lambda;\iota;m)\big[(1-\mathrm{B})w_n^{\theta}(\alpha_1;\lambda;\iota;m)-(1-\mathrm{A})\big]\Gamma(\mathrm{n}+\mathrm{s}+1)}{(\mathrm{A}-\mathrm{B})\Gamma(\mathrm{n}+1)}\sum_{i=2}^{\infty} iH(i)a_i z^{i-1},$$

where

$$H(i) = \frac{\Gamma(i)}{\Gamma(i+\mathrm{s}+1)}, \qquad (s>0\ ,i\geq 2),$$

then, for $z = re^{i\gamma}$ *and* $0<r<1$

$$\int_0^{2\pi} |D_z^{-s} f(z)|^{\mu}\, d\gamma \leq \int_0^{2\pi} |D_z^{-s} f_n(z)|^{\mu}\, d\gamma, \quad (\ s>0, \mu>0)$$

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