Journal of
Mathematics
and Applications
No 31, pp 5-16 (2009)

On generalization of some classes of Sălăgean-type multivalent harmonic functions

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Submitted by:

Abstract: In the present paper, we make generalization of the classes in [7] of Sălăgean-Type multivalent harmonic functions. We introduce sufficient coefficient condition for the class $\mathcal{H}_p^i(n; \lambda, \beta, m)$ and this condition be also necessary if certain restriction is imposed on the coefficients of these harmonic functions. Also we have obtained a representation theorem, inclusion relations and distortion bounds for these functions

AMS Subject Classification: *30C45*
Key Words and Phrases: *Multivalent harmonic functions, Sălăgean derivative operator, Inclusion relations, Distortion bounds, Representation theorem*

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$, if u and v are real harmonic. If Ω be any simply connected domain and $\Omega \subset \mathbb{C}$, then $f = h + \overline{g}$, where h and g are analytic in Ω, h is analytic part and g is co-analytic part of $f \cdot |g'(z)| < |h'(z)|$ if and only if f is locally univalent and sense preserving in Ω, see [3], [5]. Denote by

$\mathcal{H} = \{f : f = h + \overline{g}, f$ is harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}\}$.

So $f = h + \overline{g} \in \mathcal{H}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$.

Ahuja and Jahangiri [1] defined the class $\mathcal{H}_p(n)$ $(p, n \in I\!N = \{1, 2, 3, \cdots\})$ consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are sense-preserving U, and h, g are of the form

$$h(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad g(z) = \sum_{k=n+p-1}^{\infty} b_k z^k, \quad |b_{n+p-1}| < 1. \tag{1}$$

Let $f = h + \overline{g}$ given by (1), the modified Sălăgean operator of f is defined as:

$$D^i f(z) = D^i h(z) + (-1)^i \overline{D^i g(z)}, \quad p > i, \quad i \in I\!N_0 = \{0, 1, 2, \cdots\},$$

where $D^i h(z) = p^i z^p \sum\limits_{k=n+p}^{\infty} k^i a_k z^k$ and $D^i g(z) = \sum\limits_{k=n+p-1}^{\infty} k^i b_k z^k$ (see [4], [6]).

Let $\mathcal{H}_p^i(n)$ be a subclass consisting of harmonic functions $f_i = h + \overline{g}_i$, so that h and g_i are of the form:

$$h(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \quad g_i(z) = (-1)^i \sum_{k=n+p-1}^{\infty} b_k z^k, \quad \text{for} \quad a_k, b_k \geq 0, |b_{n+p-1}| < 1. \tag{2}$$

A function f in $\mathcal{H}_p(n)$ is said to be in the class $\mathcal{H}_p^i(n; \lambda, \beta, m)$ if

$$Re\left\{(1-\lambda)\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p} + \lambda(1-m)\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p} + \lambda m \frac{D^{i+2} f(z)}{\frac{\partial^{i+2}}{\partial\theta^{i+2}} z^p}\right\} > \frac{\beta}{p^{i+1}}, \tag{3}$$

where $0 \leq \beta < p, \lambda \geq 0, 0 \leq m \leq 1, p \geq i$ and $z = re^{\mathrm{i}\theta} \in U$.

As λ changes from 0 to 1, the family $\mathcal{H}_p^i(n; \lambda, \beta, m)$ provides a passage from the class of Sălăgean-type multivalent harmonic functions $\mathcal{H}_p^i R(n; \beta) \equiv \mathcal{H}_p^i(n; 0, \beta, m)$ consisting of functions f, where

$$Re\left\{\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p}\right\} > \frac{\beta}{p^{i+1}}$$

and this class was studied in [7].

To the class of Sălăgean-type multivalent harmonic functions $\mathcal{H}_p^i S(n; \beta, m) \equiv \mathcal{H}_p^i(n; 1, \beta, m)$ consisting of functions f, where

$$Re\left\{(1-m)\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p} + m\frac{D^{i+2} f(z)}{\frac{\partial^{i+2}}{\partial\theta^{i+2}} z^p}\right\} > \frac{\beta}{p^{i+1}},$$

to the class of Sălăgean-type multivalent harmonic functions (if $m = 0$) $\mathcal{H}_p^i T(n; \beta) \equiv \mathcal{H}_p^i(n; 1, \beta, 0)$ consisting of functions f satisfying

$$Re\left\{\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p}\right\} > \frac{\beta}{p^{i+1}},$$

and this class was studied in [7].

If $m = 0$, then the class $\mathcal{H}_p^i(n; \lambda, \beta, m)$ reduces to the class $\mathcal{H}_p^i U(n; \lambda, \beta) \equiv \mathcal{H}_p^i(n; \lambda, \beta, 0)$ consisting of functions f such that

$$Re\left\{(1-\lambda)\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p} + \lambda\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p}\right\} > \frac{\beta}{p^{i+1}},$$

and this class was studied in [7].

Now, we define the subclass $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \equiv \mathcal{H}_p^i(n;\lambda,\beta,m) \cap \mathcal{H}_p^i(n)$. If $m=0$ and $i=0$, then the class $\mathcal{H}_p^i(n;\lambda,\beta,m)$ reduces to the class $\mathcal{H}_pV(n;\lambda,\beta) \equiv \mathcal{H}_p^0(n;\lambda,\beta,0)$ that was studied in [2].

2. Representation Theorem

In the following theorem, we find a coefficient bound for functions in $\mathcal{H}_p^i(n;\lambda,\beta,m)$.

Theorem 1 : Let $f = h + \overline{g}$ be given by (1). Then $f \in \mathcal{H}_p^i(n;\lambda,\beta,m)$ if

$$\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i|a_k| + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i|b_k| \leq p^{i+1}-\beta, \quad (4)$$

where $0 \leq \beta < p, \lambda \geq 0, 0 \leq m \leq 1, p \geq i$ and $z = re^{\mathrm{i}\theta} \in U$.

Proof : By using the fact $Re\ \alpha \geq 0$ if and only if $|1+\alpha| \geq |1-\alpha|$ in U, it suffices to show that

$$|p^{i+1} - \beta + p^{i+1}w| \geq |p^{i+1} + \beta - p^{i+1}w|,$$

where

$$w = (1-\lambda)\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i} z^p} + \lambda(1-m)\frac{D^{i+1} f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}} z^p} + \lambda m \frac{D^{i+2} f(z)}{\frac{\partial^{i+2}}{\partial\theta^{i+2}} z^p}.$$

Substituting for h and g in w we obtain

$$\begin{aligned} w &= 1 + \sum_{k=n+p}^{\infty} [1 + (\frac{k}{p}-1)(m\frac{k}{p}+1)\lambda] \frac{k^i}{p^i} a_k \frac{z^k}{z^p} \\ &\quad + \sum_{k=n+p-1}^{\infty} [1 - (\frac{k}{p}+1)(1-m\frac{k}{p})\lambda](-1)^i \frac{k^i}{p^i} b_k \frac{\overline{z}^k}{z^p} \end{aligned}$$

and then we have

$$
\begin{aligned}
&|p^{i+1} - \beta + p^{i+1}w| - |p^{i+1} + \beta - p^{i+1}w| = |2p^{i+1} - \beta \\
&+ \sum_{k=n+p}^{\infty} [p + (k-p)(\frac{mk}{p} + 1)\lambda] k^i a_k \frac{z^k}{z^p} \\
&+ \sum_{k=n+p-1}^{\infty} [p - (k+p)(1 - \frac{mk}{p})\lambda](-1)^i k^i b_k \frac{\overline{z}^k}{z^p}| \\
&-|\beta + \sum_{k=n+p}^{\infty} [p + (k-p)(\frac{mk}{p} + 1)\lambda] k^i a_k \frac{z^k}{z^p} \\
&- \sum_{k=n+p-1}^{\infty} [p - (k+p)(1 - \frac{mk}{p})\lambda](-1)^i k^i b_k \frac{\overline{z}^k}{z^p}| \\
&\geq 2p^{i+1} - \sum_{k=n+p}^{\infty} |p + (k-p)(\frac{mk}{p} + 1)\lambda| k^i |a_k||z|^{k-p} \\
&- \sum_{k=n+p-1}^{\infty} |p + (k+p)(\frac{mk}{p} - 1)\lambda| k^i |b_k||z|^{k-p} \\
&- \sum_{k=n+p}^{\infty} |p + (k-p)(\frac{mk}{p} + 1)\lambda| k^i |a_k||z|^{k-p} \\
&- \sum_{k=n+p-1}^{\infty} |p + (k+p)(\frac{mk}{p} - 1)\lambda| k^i |b_k||z|^{k-p} \\
&\geq 2[(p^{i+1} - \beta) - \sum_{k=n+p}^{\infty} |p + (k-p)(\frac{mk}{p} + 1)\lambda| k^i |a_k| \\
&- \sum_{k=n+p-1}^{\infty} |p + (k+p)(\frac{mk}{p} - 1)\lambda| k^i |b_k|] \geq 0.
\end{aligned}
$$

The proof is complete.

The coefficient bound (4) given in Theorem 1 is sharp for the function

$$f(z) = z^p + \sum_{k=n+p}^{\infty} \frac{x_k}{|p + (k-p)(\frac{mk}{p} + 1)\lambda| k^i} z^k + \sum_{k=n+p-1}^{\infty} \frac{\overline{y}_k}{|p + (k+p)(\frac{mk}{p} - 1)\lambda| k^i} \overline{z}^k,$$

where $\sum_{k=n+p}^{\infty} |x_k| + \sum_{k=n+p-1}^{\infty} |y_k| = p^{i+1} - \beta$.

Theorem 2 : Let $f_i = h + \overline{g}_i$ be given by (2). Then $f_i \in \overline{\mathcal{H}}_p^i(n; \lambda, \beta, m)$ if and only if

$$\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda| k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda| k^i b_k \leq p^{i+1} - \beta. \quad (5)$$

Proof : From Theorem 1, we only want to prove the "only if" part of the theorem, since $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \subset \mathcal{H}_p^i(n;\lambda,\beta,m)$. If $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$, then, for $z = re^{\mathrm{i}\theta}$ in U we get

$$\begin{aligned} & Re\left\{(1-\lambda)\frac{D^i f(z)}{\frac{\partial^i}{\partial\theta^i}z^p} + \lambda(1-m)\frac{D^{i+1}f(z)}{\frac{\partial^{i+1}}{\partial\theta^{i+1}}z^p} + \lambda m\frac{D^{i+2}f(z)}{\frac{\partial^{i+2}}{\partial\theta^{i+2}}z^p}\right\} \\ = \quad & Re\left\{\frac{(1-\lambda)}{p^i}\left(\frac{D^i h(z) + (-1)^i\overline{D^i g_i(z)}}{\mathrm{i}^i z^p}\right)\right. \\ & +\frac{\lambda(1-m)}{p^{i+1}}\left(\frac{D^{i+1}h(z) - (-1)^i\overline{D^{i+1}g_i(z)}}{\mathrm{i}^{i+1}z^p}\right) \\ & \left.+\frac{\lambda m}{p^{i+2}}\left(\frac{D^{i+2}h(z) + (-1)^i\overline{D^{i+2}g_i(z)}}{\mathrm{i}^{i+2}z^p}\right)\right\} \\ \geq \quad & 1 - \frac{1}{p^{i+1}}\sum_{k=n+p}^{\infty}|p + (k-p)(\frac{mk}{p}+1)\lambda|k^i a_k r^{k-p} \\ & -\frac{1}{p^{i+1}}\sum_{k=n+p-1}^{\infty}|p + (k+p)(\frac{mk}{p}-1)\lambda|k^i b_k r^{k-p} \geq \frac{\beta}{p^{i+1}}. \end{aligned}$$

This inequality must hold for all $z \in U$. In particular, letting $z = r \to 1$, it yields the required condition (5).

As special cases of Theorem 2, we obtain the following corollaries :

Corollary 1 [7] : $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i R(n;\beta) \equiv \mathcal{H}_p^i R(n;\beta) \cap \mathcal{H}_p^i(n)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{pk^i}{p^{i+1}-\beta}a_k + \sum_{k=n+p-1}^{\infty}\frac{pk^i}{p^{i+1}-\beta}b_k \leq 1.$$

Corollary 2 : $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i S(n;\beta,m) \equiv \mathcal{H}_p^i S(n;\beta,m) \cap \mathcal{H}_p^i(n)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{|p + (k-p)(\frac{mk}{p}+1)|k^i}{p^{i+1}-\beta}a_k + \sum_{k=n+p-1}^{\infty}\frac{|p + (k+p)(\frac{mk}{p}-1)|k^i}{p^{i+1}-\beta}b_k \leq 1.$$

Corollary 3 [7] : $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i T(n;\beta) \equiv \mathcal{H}_p^i T(n;\beta) \cap H_p^i(n)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{k^{i+1}}{p^{i+1}-\beta}a_k + \sum_{k=n+p-1}^{\infty}\frac{k^{i+1}}{p^{i+1}-\beta}b_k \leq 1.$$

Corollary 4 [7] : $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i U(n;\lambda,\beta) \equiv \mathcal{H}_p^i U(n;\lambda,\beta) \cap H_p^i(n)$ if and only if

$$\sum_{k=n+p}^{\infty}\frac{|\lambda k + (1-\lambda)p|k^i}{p^{i+1}-\beta}a_k + \sum_{k=n+p-1}^{\infty}\frac{|\lambda k - (1-\lambda)p|k^i}{p^{i+1}-\beta}b_k \leq 1.$$

In the following theorem, we determine a representation theorem for functions in $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$.

Theorem 3 : $f_i = h + \overline{g}_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$ if and only if f_i can be expressed as

$$f_i(z) = X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_i}(z),$$

where $h_p(z) = z^p, h_k(z) = \frac{p^{i+1}-\beta}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i} z^k, (k = n+p, n+p+1, \cdots), g_{k_i}(z) = z^p + (-1)^i \frac{p^{i+1}-\beta}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i} \overline{z}^k, (k = n+p-1, n+p, \cdots), X_p \geq 0, Y_{n+p-1} \geq 0, X_p + \sum_{k=n+p}^{\infty} X_k + \sum_{k=n+p-1}^{\infty} Y_k = 1$, and $X_k \geq 0, Y_k \geq 0$, for $k = n+p, n+p+1, \cdots$.

Proof : For functions f_i of the form (2), we have

$$\begin{aligned} f_i(z) &= X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_i}(z) \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{p^{i+1}-\beta}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i} X_k z^k \\ &\quad +(-1)^i \sum_{k=n+p-1}^{\infty} \frac{p^{i+1}-\beta}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i} Y_k \overline{z}^k. \end{aligned}$$

Consequently, $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$, since by (5), we have

$$\begin{aligned} &\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k \\ &= \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i \cdot \frac{p^{i+1}-\beta}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i} |X_k| \\ &+ \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i \cdot \frac{p^{i+1}-\beta}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i} |Y_k| \\ &= (p^{i+1}-\beta) \left(\sum_{k=n+p}^{\infty} |X_k| + \sum_{k=n+p-1}^{\infty} |Y_k| \right) = (p^{i+1}-\beta)(1-X_p) \leq p^{i+1}-\beta. \end{aligned}$$

Conversely, assume $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$. Letting $X_p = 1 - \sum_{k=n+p}^{\infty} X_k - \sum_{k=n+p-1}^{\infty} Y_k$, where $X_k = \frac{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i}{p^{i+1}-\beta} a_k$ and $Y_k = \frac{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i}{p^{i+1}-\beta} b_k$, we obtain the required representation, since

$$\begin{aligned}
f_i(z) &= z^p - \sum_{k=n+p}^{\infty} a_k z^k + (-1)^i \sum_{k=n+p-1}^{\infty} b_k \overline{z}^k \\
&= z^p - \sum_{k=n+p}^{\infty} \frac{(p^{i+1}-\beta)X_k}{|p+(k-p)(\frac{mk}{p}+1)\lambda|k^i} z^k \\
&\quad +(-1)^i \sum_{k=n+p-1}^{\infty} \frac{(p^{i+1}-\beta)Y_k}{|p+(k+p)(\frac{mk}{p}-1)\lambda|k^i} \overline{z}^k \\
&= z^p - \sum_{k=n+p}^{\infty} (z^p - h_k(z))X_k - \sum_{k=n+p-1}^{\infty} (z^p - g_{k_i}(z))Y_k \\
&= \left(1 - \sum_{k=n+p}^{\infty} X_k - \sum_{k=n+p-1}^{\infty} Y_k\right) z^p + \sum_{k=n+p}^{\infty} h_k(z)X_k + \sum_{k=n+p-1}^{\infty} g_{k_i}(z)Y_k \\
&= X_p h_p(z) + \sum_{k=n+p}^{\infty} X_k h_k(z) + \sum_{k=n+p-1}^{\infty} Y_k g_{k_i}(z).
\end{aligned}$$

3. Inclusion Relations

In the following theorem, we discuss the inclusion relations between the above mentioned classes. The inclusion relations between the classes for the different values of λ are not so obvious.

Theorem 4 : For $n \in I\!N$ and $0 \le \beta < p$, we have:

(1) $\overline{\mathcal{H}}_p^i S(n;\beta,m) \subset \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m), 0 \le \lambda < 1$

(2) $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \subset \overline{\mathcal{H}}_p^i S(n;\beta,m), \lambda \ge 1$

(3) $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \subset \overline{\mathcal{H}}_p^i R(n;\beta), \lambda \ge 0$

(4) $\overline{\mathcal{H}}_p^i(n;\lambda,\beta,m) \subset \overline{\mathcal{H}}_p^i U(n;\lambda,\beta), \lambda \ge 0$

(5) $\overline{\mathcal{H}}_p^i S(n;\beta,m) \subset \overline{\mathcal{H}}_p^i R(n;\beta)$.

Proof (1) For $0 \le \lambda < 1$, we have

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k \\
&\le \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)|k^i b_k \\
&\le p^{i+1} - \beta. \quad \text{(by Corollary 2)}
\end{aligned}$$

Therefore (1) is obtained from Theorem 2.

(2) If $\lambda \geq 1$, then by Theorem 2

$$\sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)|k^i b_k$$
$$\leq \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k$$
$$\leq p^{i+1} - \beta.$$

Therefore, (2) is obtained from Corollary 2.
(3) If $\lambda \geq 0$, then by Theorem 2,

$$\sum_{k=n+p}^{\infty} pk^i a_k + \sum_{k=n+p-1}^{\infty} pk^i b_k$$
$$\leq \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k$$
$$\leq p^{i+1} - \beta.$$

Thus, (3) is obtained from Corollary 1.
(4) If $\lambda \geq 0$, then by Theorem 2,

$$\sum_{k=n+p}^{\infty} |\lambda k+(1-\lambda)p|k^i a_k + \sum_{k=n+p-1}^{\infty} |\lambda k-(1-\lambda)p|k^i b_k$$
$$= \sum_{k=n+p}^{\infty} |p+(k-p)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |(k+p)\lambda-p|k^i b_k$$

$$\leq \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)\lambda|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)\lambda|k^i b_k$$
$$\leq p^{i+1}-\beta.$$

Thus, (4) is obtained from Corollary 4.
(5) In view of Corollaries 1 and 2, since

$$\sum_{k=n+p}^{\infty} pk^i a_k + \sum_{k=n+p-1}^{\infty} pk^i b_k$$
$$\leq \sum_{k=n+p}^{\infty} |p+(k-p)(\frac{mk}{p}+1)|k^i a_k + \sum_{k=n+p-1}^{\infty} |p+(k+p)(\frac{mk}{p}-1)|k^i b_k.$$

The result follows.

4. Distortion Bounds

We introduce a distortion theorem for functions in $\overline{\mathcal{H}}_p^i(n;\beta,\lambda,m)$.

Theorem 5 : If $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$, $\lambda \geq 1$ and $|z| = r < 1$, then

$$|f_i(z)| \leq (1+b_{n+p-1}r^{n-1})r^p + \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} b_{n+p-1}\right) r^{n+p}$$

and

$$|f_i(z)| \geq (1-b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} b_{n+p-1}\right) r^{n+p}.$$

Proof : We prove the left hand side inequality for $|f_i|$. Let $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m)$, then by Theorem 2, we obtain:

$$\begin{aligned}
|f_i(z)| &= \left| z^p + (-1)^i b_{n+p-1}\overline{z}^{n+p-1} + \sum_{k=n+p}^{\infty}(a_k z^k + (-1)^i b_k \overline{z}^k) \right| \\
&\geq r^p - b_{n+p-1}r^{n+p-1} - \frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} \times \\
&\quad \sum_{k=n+p}^{\infty}\left(\frac{p+n(\frac{mn}{p}+m+1)\lambda}{p^{i+1}-\beta}a_k + \frac{p+n(\frac{mn}{p}+m+1)\lambda}{p^{i+1}-\beta}b_k \right)(n+p)^i r^k \\
&\geq r^p - b_{n+p-1}r^{n+p-1} - \frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} \times \\
&\quad \sum_{k=n+p}^{\infty}\left(\frac{p+(k-p)(\frac{mk}{p}+1)\lambda}{p^{i+1}-\beta}a_k + \frac{(k+p)(1-\frac{mk}{p})\lambda - p}{p^{i+1}-\beta}b_k \right)k^i r^k \\
&\geq (1-b_{n+p-1}r^{n-1})r^p - \frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} \times \\
&\quad \left[1 - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i}{p^{i+1}-\beta}b_{n+p-1} \right] r^{n+p} \\
&\geq (1-b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i} \right. \\
&\quad \left. - \frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i}b_{n+p-1} \right) r^{n+p}.
\end{aligned}$$

The proof for the right hand side inequality can be done using similar arguments and this completes the proof of theorem.

The following result follows from the left hand side inequality in Theorem 5.

Corollary 5 : If $f_i \in \overline{\mathcal{H}}_p^i(n;\lambda,\beta,m), \lambda \geq 1$, then the set

$$\begin{aligned}
&\{w : |w| < [(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i - p^{i+1} + \beta - [(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i \\
&+[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))\lambda - p](n+p-1)^i]b_{n+p-1}]/[(p+n(\frac{mn}{p}+m+1)\lambda)(n+p)^i]\}
\end{aligned}$$

is included in $f_i(U)$.

By using arguments similar to those given in the proof of Theorem 5, we get the following corollaries.

Corollary 6 [7] : If $f_i \in \overline{\mathcal{H}}_p^i R(n;\beta)$, then

$$|f_i(z)| \leq (1+b_{n+p-1}r^{n-1})r^p + \left(\frac{p^{i+1}-\beta}{p(n+p)^i} + \frac{(n+p-1)^i}{(n+p)^i}b_{n+p-1} \right) r^{n+p},$$

and

$$|f_i(z)| \geq (1-b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{p(n+p)^i} + \frac{(n+p-1)^i}{(n+p)^i}b_{n+p-1}\right)r^{n+p}.$$

Corollary 7 [7] : If $f_i \in \overline{\mathcal{H}}_p^i T(n;\beta)$, then

$$|f_i(z)| \leq (1+b_{n+p-1}r^{n-1})r^p + \left(\frac{p^{i+1}-\beta}{(n+p)^{i+1}} - \frac{(n+p-1)^{i+1}}{(n+p)^{i+1}}b_{n+p-1}\right)r^{n+p},$$

and

$$|f_i(z)| \geq (1-b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{(n+p)^{i+1}} - \frac{(n+p-1)^{i+1}}{(n+p)^{i+1}}b_{n+p-1}\right)r^{n+p}.$$

Corollary 8 [7] : If $f_i \in \overline{\mathcal{H}}_p^i U(n;\lambda,\beta)$, then

$$\begin{aligned}|f_i(z)| \leq\ & (1+b_{n+p-1}r^{n-1})r^p \\ & + \left(\frac{p^{i+1}-\beta}{(\lambda n+p)(n+p)^i} - \frac{[\lambda(n+2p-1)-p](n+p-1)^i}{(\lambda n+p)(n+p)^i}b_{n+p-1}\right)r^{n+p},\end{aligned}$$

and

$$\begin{aligned}|f_i(z)| \geq\ & (1-b_{n+p-1}r^{n-1})r^p \\ & - \left(\frac{p^{i+1}-\beta}{(\lambda n+p)(n+p)^i} - \frac{[\lambda(n+2p-1)-p](n+p-1)^i}{(\lambda n+p)(n+p)^i}b_{n+p-1}\right)r^{n+p}.\end{aligned}$$

Corollary 9 : If $f_i \in \overline{\mathcal{H}}_p^i S(n;\beta,m)$, then

$$\begin{aligned}|f_i(z)| \leq\ & (1+b_{n+p-1}r^{n-1})r^p + \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1))(n+p)^i}\right. \\ & \left. -\frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))-p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1))(n+p)^i}b_{n+p-1}\right)r^{n+p},\end{aligned}$$

and

$$\begin{aligned}|f_i(z)| \geq\ & (1-b_{n+p-1}r^{n-1})r^p - \left(\frac{p^{i+1}-\beta}{(p+n(\frac{mn}{p}+m+1))(n+p)^i}\right. \\ & \left. -\frac{[(n+2p-1)(1-\frac{1}{p}(m(n+p-1)))-p](n+p-1)^i}{(p+n(\frac{mn}{p}+m+1))(n+p)^i}b_{n+p-1}\right)r^{n+p}.\end{aligned}$$

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