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# Some Applications of a New Class of Univalent Functions Defined by Subordination Property

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## ABSTRACT

In this paper, we introduce and study some applications of a new class of univalent functions defined by subordination property. Coefficient inequality, convex linear combinations, growth and distortion bounds, radii of starlikeness, convexity and close-to- convexity and Hadamard product (or convolution) are given.

## Indexing terms/Keywords:

Univalent function; Subordination property; Hadamard product (or convolution); neighborhood; distortion bounds; convex linear combinations.

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## 1. INTRODUCTION

Let $\Psi(n)$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \ , \ (n \in \mathbb{N}) \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

If $f \in \Psi(n)$ is given by $(1.1)$ and $g \in \Psi$ is given by

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \ , \ (n \in \mathbb{N}). \tag{1.2}$$

The Hadamard product (or convolution) $(f * g)(z)$ of $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

We shall need the integral operator due to Jung – Kim – Srivastava, (see[7],[8],[10]).

$$I(z) = Q_\gamma^\tau f(z) = \binom{\tau+\gamma}{\gamma} \frac{\tau}{z^\gamma} \int_0^z t^{\gamma-1} \left(1 - \frac{t}{z}\right)^{\tau-1} f(t)\, dt \ , \tag{1.4}$$

$(\tau > 0, \gamma > -1, z \in U)$.

It can be easily verified that

$$I(z) = Q_\gamma^\tau f(z) = z - \sum_{k=n+1}^{\infty} \frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)} a_k z^k. \tag{1.5}$$

Let $\wp(n)$ denote the subclass of $\Psi(n)$ consisting of functions $f$ of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \ , \ (a_k \geq 0; n \in \mathbb{N}) \ , \tag{1.6}$$

which are analytic and univalent in $U$.

### Definition 1.1 [6]

Let $f$ and $g$ be analytic in the unite disk $U$. Then $g$ is said to be subordinate to $f$ , written $g \prec f$ or $g(z) \prec f(z)$, if there exists a Schwarz function $w$ , which is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 \ (z \in U)$, such that $g(z) = f\big(w(z)\big) \ (z \in U)$. Indeed it is known that

$g(z) \prec f(z) \ (z \in U) \Rightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$

In particular , if the function $f$ is univalent in $U$, we have the following equivalence ([8],[9]):

$g(z) \prec f(z) \ (z \in U) \Leftrightarrow g(0) = f(0) \text{ and } g(U) \subset f(U).$

### Definition 1.2

Let $Q(A, B, \alpha, n)$ consist of all analytic functions $m$ in $U$ for which

$m(0)=2$

and

$$m(z)\prec\frac{1+[B+\alpha((1-\alpha)+(A-B))]z}{1+Bz},$$

where $-1\le B<A\le 1,\ 0<A\le 1,0<\alpha\le 1$.

### Definition 1.3

For $A,B$ fixed, $-1\le B<A\le 1, 0<A\le 1, 0<\alpha\le 1,\ z\in U$, let $\Re_n(A,B,\alpha,\gamma,\tau)$ denote the class of functions $f\in\wp(n)$ of the form (1.5) for which

$$\frac{zI'(z)}{I(z)}\in Q(A,B,\alpha,n)\quad and$$

$$\frac{zI'(z)}{I(z)}\prec\frac{1+[B+\alpha((1-\alpha)+(A-B))]z}{1+Bz}\quad ,z\in U \tag{1.7}$$

where $\prec$ denotes subordination.

From the definition, it follows that $f\in\Re_n(A,B,\alpha,\gamma,\tau)$ if there exists a function $w(z)$ analytic in $U$ and satisfies $w(0)=0$ and $|w(z)|<1$ for $z\in U$, such that

$$\frac{zI'(z)}{I(z)}=\frac{1+[B+\alpha((1-\alpha)+(A-B))]w(z)}{1+Bw(z)}\quad ,\quad z\in U. \tag{1.8}$$

This condition (1.7) is equivalent to

$$\left|\frac{\frac{zI'(z)}{I(z)}-1}{B+\alpha((1-\alpha)+(A-B))-B\frac{zI'(z)}{I(z)}}\right|<1\quad ,z\in U. \tag{1.9}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [4], Ruscheweyh [9], Darwish [3], Miller and Mocanu [6] and Atshan and Kulkarni [1], but for meromorphic function studied by Atshan et al. [2] and Liu and Srivastava [5], we define the $(n,\delta)$-neighborhood of a function $f\in\wp(n)$ by

$$N_{n,\delta}(f)=\left\{g\in\wp:g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k\ and\ \sum_{k=n+1}^{\infty}k|a_k-b_k|\le\delta\right\}. \tag{1.10}$$

In particular, for the identity function $e(z)=z$, we have

$$N_{n,\delta}(e)=\left\{g\in\wp:g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k\ and\ \sum_{k=n+1}^{\infty}k|b_k|\le\delta\right\}. \tag{1.11}$$

### Definition1.4

A function $f\in\wp(n)$ is said to be in the class $\Re^{\eta}(A,B,\alpha,\gamma,\tau)$ if there exists a function $g\in\Re_n(A,B,\alpha,\gamma,\tau)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\eta\ ,\qquad (z\in U, 0\le\eta<1). \tag{1.12}$$

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## 2. COEFFICIENT INEQUALITY

First, in the following theorem, we obtain a necessary and sufficient condition for a function $f$ to be in the class $\mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$.

### Theorem 2.1

Let $f \in \wp(n)$. Then the function $f \in \mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$ if and only if

$$\sum_{k=n+1}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]a_k \le \alpha\big((1-\alpha)+(A-B)\big), \quad (2.1)$$

for

$$(-1 \le B < A \le 1, 0 < A \le 1, 0 < \alpha \le 1, \tau > 0, \gamma > -1).$$

The result is sharp with the extremal function $f$ given by

$$f(z) = z - \frac{\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]} z^{n+1}, n \in \mathbb{N} \quad (2.2)$$

**Proof:** Assume that the inequality (2.1) holds true and $|z| = 1$. Then we have

$$|zI'(z) - I(z)| - \left|I(z)\left[B+\alpha\big((1-\alpha)+(A-B)\big)\right] - BzI'(z)\right|$$

$$= \left|\left(z - \sum_{k=n+1}^{\infty} ka_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\right) - \left(z - \sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\right)\right|$$

$$- \left|\left(z - \sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\right)\left[B+\alpha\big((1-\alpha)+(A-B)\big)\right] - B\left(z - \sum_{k=n+1}^{\infty} ka_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\right)\right|$$

$$= \left|-\sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k(k-1)\right| -$$

$$\left|z\left[B+\alpha\big((1-\alpha)+(A-B)\big)\right] - \sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\left[B+\alpha\big((1-\alpha)+(A-B)\big)\right] - Bz + B\sum_{k=n+1}^{\infty} ka_kz^k\right|$$

$$= \left|-\sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k(k-1)\right|$$

$$-\Bigg|z\big([B+\alpha\big((1-\alpha)+(A-B)\big)]-B\big)$$
$$-\sum_{k=n+1}^{\infty} a_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]^k \big([B+\alpha\big((1-\alpha)+(A-B)\big)]-B\big)\Bigg|$$
$$\leq \sum_{k=n+1}^{\infty} a_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right](k-1)-\alpha\big((1-\alpha)+(A-B)\big)$$
$$+\sum_{k=n+1}^{\infty} a_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\Big(\alpha\big((1-\alpha)+(A-B)\big)\Big)$$
$$=\sum_{k=n+1}^{\infty} \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\big[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\big]a_k$$
$$-\alpha\big((1-\alpha)+(A-B)\big)\leq 0,$$

by hypothesis. Thus by maximum modulus Theorem, $f\in\Re_n(A,B,\alpha,\gamma,\tau)$.

Conversely, suppose that $f\in\Re_n(A,B,\alpha,\gamma,\tau)$. Then from(1.9), we have

$$\left|\frac{\frac{z(I(z))'}{I(z)}-1}{B+\alpha\big((1-\alpha)+(A-B)\big)-B\frac{z(I(z))'}{I(z)}}\right|$$

$$=\left|\frac{z(I(z))'-I(z)}{B+\alpha\big((1-\alpha)+(A-B)\big)I(z)-Bz(I(z))'}\right|$$
$$=\left|\frac{-\sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k(k-1)}{[B+\alpha((1-\alpha)+(A-B))]\left(z-\sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\right)-B\left(z-\sum_{k=n+1}^{\infty} ka_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\right)}\right|<1.$$

Since $|Re(z)|\leq|z|$ for all, we have

$$Re\left\{\frac{\sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k(k-1)}{[B+\alpha((1-\alpha)+(A-B))]\left(z-\sum_{k=n+1}^{\infty} a_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\right)-B\left(z-\sum_{k=n+1}^{\infty} ka_k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]z^k\right)}\right\}<1. \qquad (2.3)$$

We choose the value of $z$ on the real axis so that $\frac{zI'(z)}{I(z)}$ is real. Upon clearing the denominator of (2.3) and letting $z\to 1$ through real values, so we can write (2.3) as

$$\sum_{k=n+1}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\big[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\big]a_k\leq\alpha\big((1-\alpha)+(A-B)\big).$$

### Corollary 2.1

Let the function $f$ of the form (1.6) be in the class $\Re_n(A,B,\alpha,\gamma,\tau)$.Then

$$a_k\leq\frac{\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\big[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\big]}\ ,(k\geq n+1,n\in\mathbb{N}),(2.4)$$

where the equality holds true for the function (2.2).

**Proof:** The result (2.4) follows from Theorem (2.1).

## 3. INCLUSION THEOREMS

We give some interesting properties of the class $\Re_n(A,B,\alpha,\gamma,\tau)$.

### Theorem 3.1

Let $-1 \le B < A \le 1$, $-1 \le \acute{B} < \acute{A} \le 1$ and $0 < \alpha \le 1$. Then

$$\Re_n(A,B,\alpha,\gamma,\tau) = \Re_n(\acute{A},\acute{B},\alpha,\gamma,\tau). \tag{3.1}$$

If and only if

$$\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)} = \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)\right]}{\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)}. \tag{3.2}$$

**Proof:** Let $f \in \Re_n(A,B,\alpha,\gamma,\tau)$. and (3.2) hold true. Then by Theorem (2.1), we have

$$\sum_{k=n+1}^{\infty}\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)\right]}{\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)}a_k = \sum_{k=n+1}^{\infty}\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)}a_k \le 1.$$

This implies $f \in \Re_n(\acute{A},\acute{B},\alpha,\gamma,\tau)$. Similarly it can be shown that $f \in \Re_n(\acute{A},\acute{B},\alpha,\gamma,\tau)$ implies $f \in \Re_n(A,B,\alpha,\gamma,\tau)$ . Hence (3.2) implies $\Re_n(A,B,\alpha,\gamma,\tau) = \Re_n(\acute{A},\acute{B},\alpha,\gamma,\tau)$. Conversely, suppose (3.1) holds true. Notice that a function defined by (1.6) belonging to $\Re_n(A,B,\alpha,\gamma,\tau)$ will belong to $\Re_n(\acute{A},\acute{B},\alpha,\gamma,\tau)$ only if

$$\sum_{k=n+1}^{\infty}\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)\right]}{\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)}a_k \le \sum_{k=n+1}^{\infty}\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)}a_k$$

that is if

$$\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)\right]}{\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)} \le \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)} \tag{3.3}$$

Similarly, we can show that

$$\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A-B)\right)\right]}{\alpha\left((1-\alpha)+(A-B)\right)}$$
$$\leq \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)\right]}{\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)}. \qquad (3.4)$$

(3.3)and (3.4) together imply (3.2). Hence the result.

We state some interesting deduction which follow using Theorem (2.1) and Theorem (2.2).

### Theorem 3.2

Let $-1 \leq B < A_1 \leq A_2 \leq 1$. Then

$$\Re_n(A_1, B, \alpha, \gamma, \tau) \supseteq \Re_n(A_2, B, \alpha, \gamma, \tau).$$

**Proof:** Notice that

$$\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A_1-B)\right)\right]}{\alpha\left((1-\alpha)+(A_1-B)\right)} \leq \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A_2-B)\right)\right]}{\alpha\left((1-\alpha)+(A_2-B)\right)}, \text{ for } A_1 \leq A_2, \quad (3.5)$$

if $f \in \Re_n(A_2, B, \alpha, \gamma, \tau)$ , we have

$$\sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A_1-B)\right)\right]}{\alpha\left((1-\alpha)+(A_1-B)\right)} a_k$$
$$\leq \sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A_2-B)\right)\right]}{\alpha\left((1-\alpha)+(A_2-B)\right)} a_k \leq 1.$$

Thus by Theorem (2.1) it follows that $f \in \Re_n(A_1, B, \alpha, \gamma, \tau)$. ■.

### Theorem 3.3

Let $-1 \leq B_1 \leq B_2 < A \leq 1$. Then

$$\Re_n(A, B_1, \alpha, \gamma, \tau) \subseteq \Re_n(A, B_2, \alpha, \gamma, \tau) .$$

**Proof:** Notice that

$$\frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A-B_1)\right)\right]}{\alpha\left((1-\alpha)+(A-B_1)\right)} \leq \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A-B_2)\right)\right]}{\alpha\left((1-\alpha)+(A-B_2)\right)}, for\ B_1 \leq B_2, \quad (3.6)$$

if $f \in \Re_n(A, B_2, \alpha, \gamma, \tau)$ , we have

$$\sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A-B_1)\right)\right]}{\alpha\left((1-\alpha)+(A-B_1)\right)} a_k$$
$$\leq \sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\left((1-\alpha)+(A-B_2)\right)\right]}{\alpha\left((1-\alpha)+(A-B_2)\right)} a_k \leq 1.$$

Thus by Theorem (2.1) it follows that $f \in \Re_n(A, B_1, \alpha, \gamma, \tau)$.

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## 4. GROWTH AND DISTORTION BOUNDS

We now state the following growth and distortion inequalities for the class $\mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$.

### Theorem 4.1

Let the function $f$ defined by (1.6) be in the class $\mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$. Then

$$\big||f(z)|-|z|\big| \leq \frac{\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+n+1)\Gamma(\gamma+1)}\right]\left[n+\alpha\big((1-\alpha)+(A-B)\big)\right]}|z|^{n+1}, \qquad (4.1)$$

$n \in \mathbb{N}$

and

$$\big||f'(z)|-1\big| \leq \frac{(n+1)\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+n+1)\Gamma(\gamma+1)}\right]\left[n+\alpha\big((1-\alpha)+(A-B)\big)\right]}|z|^{n}, \qquad (4.2)$$

$n \in \mathbb{N}$.

The result in (4.1) and (4.2) are sharp with the extremal function

$$f(z)=z-\frac{\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+n+1)\Gamma(\gamma+1)}\right]\left[n+\alpha\big((1-\alpha)+(A-B)\big)\right]}|z|^{n+1}, n \in \mathbb{N}.$$

**Proof:** We have

$$f(z)=z-\sum_{k=n+1}^{\infty} a_k z^k,$$

therefore,

$$|f(z)| \leq |z|+\sum_{k=n+1}^{\infty} a_k |z|^k \leq |z|+|z|^{n+1}\sum_{k=n+1}^{\infty} a_k \qquad (4.3)$$

$$\leq |z|+\frac{\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[n+\alpha\big((1-\alpha)+(A-B)\big)\right]}|z|^{n+1}.$$

Similarly

$$|f(z)| \geq |z|-\sum_{k=n+1}^{\infty} a_k |z|^k \geq |z|-|z|^{n+1}\sum_{k=n+1}^{\infty} a_k \qquad (4.4)$$

$$\geq |z|-\frac{\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+n+1)\Gamma(\gamma+1)}\right]\left[n+\alpha\big((1-\alpha)+(A-B)\big)\right]}|z|^{n+1}.$$

Combining (4.3) and (4.4) we get the result (4.1).

The next result in (4.2) can be derived similarly.

## 5. CONVEX LINEAR COMBINATIONS

Now, we state a theorem of convex linear combinations of the functions in the class $\mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$.

### Theorem 5.1

Let the function

$$f_j(z) = z - \sum_{k=n+1}^{\infty} a_{k,j}\, z^k \quad ,(a_{k,j} \geq 0\ , j = 1,2,\dots,l)$$

be in the class $\mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$. Then

$$y(z) = \sum_{j=1}^{l} c_j\, f_j(z)\ \in \mathfrak{R}(A,B,\alpha,\gamma,\tau),$$

where

$$\sum_{j=1}^{l} c_j = 1 \ \text{ and } \ c_j \geq 0\ \ (j = 1,2,\dots,l).$$

Thus, we note that $\mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$ is a convex set.

**Proof:** We have

$$y(z) = \sum_{j=1}^{l} c_j\ (z - \sum_{k=n+1}^{\infty} a_{k,j}\, z^k) \tag{5.1}$$

$$= z\sum_{j=1}^{l} c_j - \sum_{j=1}^{l}\sum_{k=n+1}^{\infty} c_j\ a_{k,j}\, z^k$$

$$= z - \sum_{k=n+1}^{\infty}\left(\sum_{j=1}^{l} a_{k,j}\, c_j\right) z^k$$

$$= z - \sum_{k=n+1}^{\infty} e_k\, z^k \text{ ,where } e_k = \sum_{j=1}^{l} a_{k,j}\, c_j\ .$$

Since $f_j \in \mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$ by (2.1), we have

$$\sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)}\, a_{k,j} \leq 1. \tag{5.2}$$

In view of (5.2), $y(z) \in \mathfrak{R}_n(A,B,\alpha,\gamma,\tau)$ if

$$\sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)}\, e_k \leq 1.$$

Now, we have

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$$\sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)} e_k$$

$$= \sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)} \sum_{j=1}^{l} a_{k,j}\, c_j$$

$$= \sum_{j=1}^{l} c_j \sum_{k=n+1}^{\infty} \frac{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\alpha\big((1-\alpha)+(A-B)\big)} a_{k,j}$$

$$\leq \sum_{j=1}^{l} c_j = 1\ .$$

Thus $y(z) \in \Re_n(A,B,\alpha,\gamma,\tau)$.

## 6. THE NEIGHBORHOOD PROPERTY

In the following theorem, we determine the neighborhood property for the class $\Re^{\eta}(A,B,\alpha,\gamma,\tau)$ .

### Theorem 6.1

Let $g \in \Re_n(A,B,\alpha,\gamma,\tau)$ and

$$\eta = 1 - \frac{\delta}{n+1}\ \frac{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[\big((n+1)-1\big)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+n+1)\Gamma(\gamma+1)}\right]\left[\big((n+1)-1\big)+\alpha\big((1-\alpha)+(\acute{A}-\acute{B})\big)\right]-\alpha\big((1-\alpha)+(A-B)\big)}\ . \qquad (6.1)$$

Then $N_{n,\delta}(g) \subset \Re_n(A,B,\alpha,\gamma,\tau)$

**Proof:** Assume that $f \in N_{n,\delta}(g)$. We want to find from (1.10) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta\ ,$$

which readily implies the following coefficient inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1}\ , (n \in \mathbb{N}). \qquad (6.2)$$

Next, since $g \in \Re_n(A,B,\alpha,\gamma,\tau)$, in view of Theorem (2.1) such that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[n+\alpha\big((1-\alpha)+(A-B)\big)\right]},$$

we have

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\alpha\big((1-\alpha)+(A-B)\big)}{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+n+1)\Gamma(\gamma+1)}\right]\left[\big((n+1)-1\big)+\alpha\big((1-\alpha)+(A-B)\big)\right]}. \qquad (6.3)$$

Using (6.2) and (6.3) , we get

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$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1-\sum_{k=n+1}^{\infty} b_k} \leq$$

$$\frac{\delta}{n+1}\ \frac{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+n+1)\Gamma(\gamma+1)}\right]\left[((n+1)-1)+\alpha\big((1-\alpha)+(A-B)\big)\right]}{\left[\frac{\Gamma(\gamma+n+1)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+n+1)\Gamma(\gamma+1)}\right]\left[((n+1)-1)+\alpha\left((1-\alpha)+(\acute{A}-\acute{B})\right)\right]-\alpha((1-\alpha)+(A-B))},$$

provided that $\eta$ is given by(6.1). Thus by condition (1.12) $f \in \Re_n(A,B,\alpha,\gamma,\tau)$.

## 7. SUBORDINATION PROPERTY

### Theorem 7.1

For $n=1$, let $f \in \Re_1(A,B,\alpha,\gamma,\tau)$ and $g$ be an arbitrary element of $\wp(1)$ such that $g \prec f$, defined in Definition (1.1) , and if

$$g_k = \frac{1}{k!}\left[\frac{d^k(f(w(z))}{dz^k}\right]_{z=0} \qquad (7.1)$$

also if

$$\frac{\sum_{k=2}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\left[(k-1)+\alpha\big((1-\alpha)+(A_1-B)\big)\right]|g_k|}{|g_1|} \leq \alpha\big((1-\alpha)+(A_1-B)\big). \qquad (7.2)$$

Then $g \in \Re_1(A,B,\alpha,\gamma,\tau)$.

**Proof:** Since $g \prec f$ by definition of subordination there is analytic function $w(z)$ such that $|w(z)| \leq |z|$ and $g(z)=f(w(z))$. But $g$ is the composition of two analytic functions in the unit disk , therefore we can expand this function in terms of Taylor series at origin as below

$$g(z)=\sum_{k=0}^{\infty} g_k z^k \ ,$$

where $g_k$ is defined in (7.1). Hence

$$g_0 = \frac{f(w(0))}{0!} = 0 \ , g_1 = \frac{w'(0)f'(0)}{1!} = w'(0).$$

Therefore, we can write

$$g(z) = g_1 z - \sum_{k=2}^{\infty} g_k z^k$$

and

$$Ig(z) = Q_\gamma^\tau g(z) = g_1 z - \sum_{k=2}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] g_k z^k \ ,$$

we must prove $g \in \Re_1(A,B,\alpha,\gamma,\tau)$, in other words , we show that

$$\left| \frac{\frac{z(Ig(z))'}{Ig(z)} - 1}{B + \alpha\big((1-\alpha) + (A-B)\big) - B\,\frac{z(Ig(z))'}{Ig(z)}} \right|$$

$$= \left| \frac{z(Ig(z))' - Ig(z)}{B + \alpha\big((1-\alpha) + (A-B)\big)Ig(z) - Bz(Ig(z))'} \right|$$

$$= \left| \frac{-\sum_{k=n+1}^{\infty} g_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] z^k (k-1)}{\big[B+\alpha\big((1-\alpha)+(A-B)\big)\big]\left(g_1 z - \sum_{k=n+1}^{\infty} g_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] z^k\right) - B\left(g_1 z - \sum_{k=n+1}^{\infty} k g_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] z^k\right)} \right| < 1\,.$$

Since $|Re(z)| \leq |z|$ for all z, we have

$$Re\left\{ \frac{\sum_{k=n+1}^{\infty} g_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] z^k (k-1)}{\big[B+\alpha\big((1-\alpha)+(A-B)\big)\big]\left(g_1 z - \sum_{k=n+1}^{\infty} g_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] z^k\right) - B\left(g_1 z - \sum_{k=n+1}^{\infty} k g_k \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] z^k\right)} \right\} < 1\,. \quad (7.3)$$

We can choose value of $z$ on the real axis so that $z(Ig(z))'$ is real. Let $z \to 1^-$ through real values, so we can write (7.3) as

$$\sum_{k=n+1}^{\infty} \left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] \big[(k-1) + \alpha\big((1-\alpha)+(A-B)\big)\big] g_k \leq g_1 \alpha\big((1-\alpha)+(A-B)\big).$$

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