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ON A CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY GENERALIZED RUSCHEWEYH DERIVATIVES I

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Abstract

In the present paper, we have studied a class $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$ of analytic and univalent functions as defined by making use of the generalized Ruscheweyh derivatives in the unit disk U and obtain some sharp results including coefficient inequality, Radii of starlikeness, convexity and close-to-convexity, distortion theorem, extreme points, closure theorem and Hadamard product.

1. Introduction

Let W denote the class of functions analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $T(n)$ denote the subclass of W consisting of functions of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0, n \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1}$$

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which are analytic and univalent in the unit disk U. Then the function $f \in T(n)$ is said to be in the class $S(n, \rho)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho, \quad (z \in U,\ 0 \le \rho < 1). \tag{2}$$

A function $f \in S(n, \rho)$ is called *starlike function* of order ρ.

A function $f \in T(n)$ is said to be in the class $C(n, \rho)$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \quad (z \in U,\ 0 \le \rho < 1). \tag{3}$$

A function $f \in C(n, \rho)$ is called *convex function* of order ρ.

It is observed that $f \in C(n, \rho)$ if and only if

$$zf' \in S(n, \rho), \ \ \forall n \in \mathbb{N} \ [2]. \tag{4}$$

A function $f \in T(n)$ is said to be in the class $K(n, \rho)$ if there is a convex function g such that

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \rho, \quad (\forall z \in U,\ 0 \le \rho < 1). \tag{5}$$

A function $f \in K(n, \rho)$ is called *close-to-convex* of order ρ. We shall need the fractional derivative operator ([8], [9]) in this paper.

Let $a, b, c \in \mathbb{C}$ with $c \ne 0, -1, -2, \ldots$. The Gaussain hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1 \equiv {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n = 0) \\ \lambda(\lambda + 1)\cdots(\lambda + n - 1), & (n \in \mathbb{N}). \end{cases}$$

Definition 1. Let $0 \le \vartheta < 1$ and $\mu, \nu \in \mathbb{R}$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\vartheta,\mu,\nu}$ of a function f is defined by:

$$J_{0,z}^{\vartheta,\mu,\nu} f(z)$$

$$= \begin{cases} \dfrac{1}{\Gamma(1-\theta)} \dfrac{d}{dz} \left\{ z^{\vartheta-\mu} \displaystyle\int_0^z (z-\varepsilon)^{-\vartheta} f(\varepsilon) {}_2F_1\left(\mu-\theta, 1-\nu; 1-\theta; 1-\frac{\varepsilon}{z}\right) d\varepsilon \right\}, & (0 \le \theta < 1), \\ \dfrac{d^n}{dz^n} J_{0,z}^{\vartheta-n,\mu,\nu} f(z), & (n \le \theta < n+1, n \in \mathbb{N}), \end{cases} \tag{7}$$

where the function f is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^{\varepsilon}), \quad (z \to 0), \tag{8}$$

for $\varepsilon > \max\{0, \mu - \nu\} - 1,$ and the multiplicity of $(z-\varepsilon)^{-\vartheta}$ is removed by required $\log(z - \varepsilon)$ to be real when $(z - \varepsilon) > 0.$

The fractional derivative of order θ of a function f is defined by

$$D_z^{\vartheta}\{f(z)\} = \frac{1}{\Gamma(1-\theta)} \frac{d}{dz} \int_0^z \frac{f(\varepsilon)}{(z-\varepsilon)^{\vartheta}} d\varepsilon, \quad 0 \le \theta < 1, \tag{9}$$

where f is chosen as in (7), and the multiplicity of $(z-\varepsilon)^{-\vartheta}$ is removed by required $\log(z - \varepsilon)$ to be real when $(z - \varepsilon) > 0.$

By comparing (7) and (9), we find

$$J_{0,z}^{\vartheta,\vartheta,\nu} f(z) = D_z^{\vartheta}\{f(z)\}, \quad (0 \le \theta < 1). \tag{10}$$

In terms of gamma function, we have

$$J_{0,z}^{\vartheta,\mu,\nu} z^k = \frac{\Gamma(k+1)\Gamma(1-\mu+\nu+k)}{\Gamma(1-\mu+k)\Gamma(1-\theta+\nu+k)} z^{k-\mu},$$

$$(0 \le \theta < 1, \mu, \nu \in \mathbb{R} \text{ and } k > \max\{0, \mu-\nu\} - 1). \tag{11}$$

Definition 2. Let $f \in T(n)$ be given by (1). Then the class $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$ is defined by

$$T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$$

$$= \left\{ f \in T(n) : \left| \frac{\frac{z(\Im_1^{\vartheta,\mu} f(z))''}{(\Im_1^{\vartheta,\mu} f(z))'}}{2\tau\left(1 - \alpha + \frac{z(\Im_1^{\vartheta,\mu} f(z))''}{(\Im_1^{\vartheta,\mu} f(z))'}\right) - \frac{z(\Im_1^{\vartheta,\mu} f(z))''}{(\Im_1^{\vartheta,\mu} f(z))'}} \right| < \beta, \right.$$

$$\left. z \in U,\ 0 < \beta \leq 1,\ \frac{1}{2} \leq \tau \leq 1,\ 0 \leq \alpha < \frac{1}{2\tau} \text{ and } \theta > -1 \right\}, \tag{12}$$

where $\Im_1^{\vartheta,\mu} f(z)$ is a generalized Ruscheweyh derivative defined by Goyal and Goyal [3, p. 442] as

$$\Im_1^{\vartheta,\mu} f(z) = \frac{\Gamma(\mu - \theta + \nu + 2)}{\Gamma(\nu + 2)\Gamma(\mu + 1)} zJ_{0,z}^{\vartheta,\mu,\nu}(z^{\mu-1} f(z))$$

$$= z - \sum_{k=n+1}^{\infty} a_k C_1^{\vartheta,\mu}(k) z^k, \tag{13}$$

where

$$C_1^{\vartheta,\mu}(k) = \frac{\Gamma(k + \mu)\Gamma(\nu + 2 + \mu - \theta)\Gamma(k + \nu + 1)}{\Gamma(k)\Gamma(k + \nu + 1 + \mu - \theta)\Gamma(\nu + 2)\Gamma(1 + \mu)}. \tag{14}$$

For $\mu = \theta = \gamma,\ \nu = 1,$ the generalized Ruscheweyh derivatives reduce to ordinary Ruscheweyh derivatives of f of order γ [5]:

$$D^{\gamma} f(z) = \frac{z}{\Gamma(\gamma + 1)} D^{\gamma}(z^{\gamma-1} f(z)) = z - \sum_{k=n+1}^{\infty} a_k C_k(\gamma) z^k, \tag{15}$$

where

$$C_k(\gamma) = \frac{(\gamma + 1)(\gamma + 2)\cdots(\gamma + k - 1)}{(k - 1)!}. \tag{16}$$

The class $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$ contains well-known classes of analytic functions, for example:

(i) If $\mu = \theta = 0,\ \nu = 1,\ n = 1,$ then we get the class $T^{0,0,1}(1, \tau, \alpha, \beta)$ studied by Aqlan and Kulkarni [1].

(ii) If $\mu = \theta = 0,\ \nu = 1,\ \rho = \beta,\ \alpha = 0,\ \tau = \frac{1}{2},$ then we get the class of convex functions of order $\rho, (C(n, \rho))$.

The same properties have been found for other classes in [4], [6] and [7].

2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for function to be in the class $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$.

Theorem 1. *Let* $f \in T(n)$. *Then* $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$ *if and only if*

$$\sum_{k=n+1}^{\infty} k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)a_k \le 2\beta\tau(1-\alpha), \tag{17}$$

where $0 < \beta \le 1,\ \frac{1}{2} \le \tau \le 1,\ 0 \le \alpha < \frac{1}{2\tau},\ \theta > -1,\ n \in \mathbb{N}$ *and* $C_1^{\vartheta,\mu}(k)$ *is given by* (14). *The result* (17) *is sharp for the function*

$$f(z) = z - \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} z^k, \quad k \ge n+1.$$

Proof. For $|z| = 1,$ we have

$$|z(\Im_1^{\vartheta,\mu}f(z))''| - \beta|2\tau[(1-\alpha)(\Im_1^{\vartheta,\mu}f(z))' + z(\Im_1^{\vartheta,\mu}f(z))''] - z(\Im_1^{\vartheta,\mu}f(z))''|$$

$$= \left| -\sum_{k=n+1}^{\infty} k(k-1)C_1^{\vartheta,\mu}(k)a_k z^{k-1} \right|$$

$$-\beta\left| 2\tau(1-\alpha) - \sum_{k=n+1}^{\infty}[2k\tau(k-\alpha)-k(k-1)]C_1^{\vartheta,\mu}(k)a_k z^{k-1} \right|$$

$$\le \sum_{k=n+1}^{\infty} k(k-1)C_1^{\vartheta,\mu}(k)a_k - 2\tau\beta(1-\alpha) + \beta\sum_{k=n+1}^{\infty} k[2\tau(k-\alpha)-k+1]C_1^{\vartheta,\mu}(k)a_k$$

$$= \sum_{k=n+1}^{\infty} k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)a_k - 2\beta\tau(1-\alpha) \leq 0.$$

By hypothesis.

Thus by maximum modulus theorem $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Conversely, assume that

$$\left| \frac{\dfrac{z(\Im_1^{\vartheta,\mu} f(z))''}{(\Im_1^{\vartheta,\mu} f(z))'}}{2\tau\left(1-\alpha+\dfrac{z(\Im_1^{\vartheta,\mu} f(z))''}{(\Im_1^{\vartheta,\mu} f(z))'}\right)-\dfrac{z(\Im_1^{\vartheta,\mu} f(z))''}{(\Im_1^{\vartheta,\mu} f(z))'}} \right|$$

$$= \left| \frac{z(\Im_1^{\vartheta,\mu} f(z))''}{2\tau(1-\alpha)(\Im_1^{\vartheta,\mu} f(z))' + 2\tau(\Im_1^{\vartheta,\mu} f(z))'' - z(\Im_1^{\vartheta,\mu} f(z))''} \right|$$

$$= \left| \frac{-\sum_{k=n+1}^{\infty} k(k-1)C_1^{\vartheta,\mu}(k)a_k z^{k-1}}{2\tau(1-\alpha)\left(1-\sum_{k=n+1}^{\infty} kC_1^{\vartheta,\mu}(k)a_k z^{k-1}\right)+\sum_{k=n+1}^{\infty} k(k-1)[1-2\tau]C_1^{\vartheta,\mu}(k)a_k z^{k-1}} \right| < \beta.$$

Since $|\operatorname{Re}(z)| \leq |z|$, for all z, we have

$$\operatorname{Re}\left\{ \frac{\sum_{k=n+1}^{\infty} k(k-1)C_1^{\vartheta,\mu}(k)a_k z^{k-1}}{2\tau(1-\alpha)\left(1-\sum_{k=n+1}^{\infty} kC_1^{\vartheta,\mu}(k)a_k z^{k-1}\right)+\sum_{k=n+1}^{\infty} k(k-1)[1-2\tau]C_1^{\vartheta,\mu}(k)a_k z^{k-1}} \right\} < \beta. \quad (18)$$

We can choose value of z on the real axis so that $(\Im_1^{\vartheta,\mu} f(z))'$ is real.

Let $z \to 1^-$, through real values, so we can write (18) as

$$\sum_{k=n+1}^{\infty} k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)a_k \leq 2\beta\tau(1-\alpha).$$

Finally, sharpness follows if we take

$$f(z) = z - \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} z^k, \quad k = n+1, n+2, \ldots. \tag{19}$$

The proof is complete.

Corollary 1. *Let* $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. *Then*

$$a_k \leq \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}, \quad k = n+1, n+2, \ldots. \tag{20}$$

The equality in (20) *is attained for the function f given by* (19).

Theorem 2. *The class* $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$ *is convex set.*

Proof. Let *f*, *g* be the arbitrary elements of $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Then for every t $(0 < t < 1)$, we show that $(1-t)f(z) + tg(z) \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Thus, we have

$$(1-t)f(z) + tg(z) = z - \sum_{k=n+1}^{\infty} [(1-t)a_k + tb_k] z^k$$

and

$$\sum_{k=n+1}^{\infty} \left[\frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]}{2\beta\tau(1-\alpha)}\right][(1-t)a_k + tb_k]C_1^{\vartheta,\mu}(k)$$

$$= (1-t)\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]}{2\beta\tau(1-\alpha)} a_k C_1^{\vartheta,\mu}(k)$$

$$+ t\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]}{2\beta\tau(1-\alpha)} b_k C_1^{\vartheta,\mu}(k) \leq 1.$$

This completes the proof.

Remark 1. Assume that *f* and *g* are in $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Then the function *y* defined by $y(z) = \frac{1}{2}[f(z) + g(z)]$ is also in the class $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$.

3. Radii of Starlikeness, Convexity and Close-to-convexity

In the next theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$.

Theorem 3. *Let* $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. *Then* f *is a starlike of order* ρ $(0 \leq \rho < 1)$ *in* $|z| < r = r_1(\theta, \mu, \nu, \beta, \tau, \alpha, \rho)$, *where*

$$r_1(\theta, \mu, \nu, \beta, \tau, \alpha, \rho)$$

$$= \inf_k \left\{ \frac{(1-\rho)k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{(k-\rho)2\beta\tau(1-\alpha)} \right\}^{\frac{1}{k-1}}, \quad k = n+1, n+2, \ldots. \quad (21)$$

The estimate is sharp for the function

$$f(z) = z - \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} z^k, \quad k = n+1, n+2, \ldots.$$

Proof. Let $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Then by Theorem 1

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]}{2\beta\tau(1-\alpha)} a_k C_1^{\vartheta,\mu}(k) \leq 1.$$

For $0 \leq \rho < 1$, we need to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$, we have to show that

$$\left| \frac{zf'(z) - f(z)}{f(z)} \right| = \left| \frac{-\sum_{k=n+1}^{\infty}(k-1)a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k z^{k-1}} \right| \leq \frac{\sum_{k=n+1}^{\infty}(k-1)a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k |z|^{k-1}} \leq 1 - \rho.$$

Hence

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\rho}{1-\rho} \right) a_k |z|^{k-1} \leq 1.$$

This is enough to consider

$$|z|^{k-1} \leq \frac{(1-\rho)k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{(k-\rho)2\beta\tau(1-\alpha)},$$

therefore

$$|z| \leq \left\{\frac{(1-\rho)k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{(k-\rho)2\beta\tau(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{22}$$

Setting $|z| = r_1(\theta, \mu, \nu, \beta, \tau, \alpha, \rho)$ in (22), we get the radius of starlikeness, which completes the proof of Theorem 3.

By using (4), we obtain the following theorem.

Theorem 4. *Let* $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. *Then f is convex of order* ρ $(0 \leq \rho < 1)$ *in* $|z| < r = r_2(\theta, \mu, \nu, \beta, \tau, \alpha, \rho)$, *where*

$$r_2(\vartheta, \mu, \nu, \beta, \tau, \alpha, \rho) = \inf_k \left\{\frac{(1-\rho)[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{(k-\rho)2\beta\tau(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{23}$$

The estimate is sharp for the function

$$f(z) = z - \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} z^k, \quad k = n+1, n+2, \ldots.$$

Proof. Let $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Then by Theorem 1

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]}{2\beta\tau(1-\alpha)} a_k C_1^{\vartheta,\mu}(k) \leq 1.$$

For $0 \leq \rho < 1$, we show that $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\rho$, that is

$$\left|\frac{-\sum_{k=n+1}^{\infty} k(k-1)a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty} ka_k z^{k-1}}\right| \leq \frac{\sum_{k=n+1}^{\infty} k(k-1)a_k |z|^{k-1}}{1-\sum_{k=n+1}^{\infty} ka_k |z|^{k-1}} \leq 1-\rho,$$

or equivalently

$$\sum_{k=n+1}^{\infty} k\left(\frac{k-\rho}{1-\rho}\right)a_k |z|^{k-1} \leq 1.$$

It is enough letting

$$|z|^{k-1} \le \frac{(1-\rho)[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{(k-\rho)2\beta\tau(1-\alpha)}.$$

Therefore,

$$|z| \le \left\{\frac{(1-\rho)[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{(k-\rho)2\beta\tau(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{24}$$

Setting $|z| = r_2(\theta, \mu, \nu, \beta, \tau, \alpha, \rho)$ in (24), we get the radius of convexity, which completes the proof of Theorem 4.

Theorem 5. *Let* $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. *Then f is close-to-convex of order* δ, $0 \le \delta < 1$ *in* $|z| < r = r_3(\theta, \mu, \nu, \beta, \tau, \alpha, \delta)$, *where*

$$r_3(\vartheta, \mu, \nu, \beta, \tau, \alpha, \delta) = \inf_k \left\{\frac{(1-\delta)[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{25}$$

The estimate is sharp for the function

$$f(z) = z - \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} z^k, \quad k = n+1, n+2, \ldots.$$

Proof. Let $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Then by Theorem 1,

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]}{2\beta\tau(1-\alpha)} a_k C_1^{\vartheta,\mu}(k) \le 1,$$

for $0 \le \delta < 1$, we need to show that $|f'(z) - 1| \le 1 - \delta$ for $|z| < r = r_3(\theta, \mu, \nu, \beta, \tau, \alpha, \delta)$, when $r_3(\theta, \mu, \nu, \beta, \tau, \alpha, \delta)$ is given by (25). Now

$$|f'(z) - 1| = \left|\sum_{k=n+1}^{\infty} k a_k z^{k-1}\right| \le \sum_{k=n+1}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \le 1 - \delta$ if

$$\sum_{k=n+1}^{\infty} \left(\frac{k}{1-\delta}\right) a_k |z|^{k-1} \le 1$$

but, by Theorem 1 above inequality holds true if

$$|z|^{k-1} \leq \frac{(1-\delta)[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)},$$

and this completes the proof.

4. Distortion Theorem

In the next theorem, we will find distortion bounds for $\Im_1^{\vartheta,\mu} f(z)$.

Theorem 6. *Let the function* $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. *Then*

$$|z| - \frac{2\beta\tau(1-\alpha)}{(n+1)[n(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(n+1)}|z|^{n+1} \leq |f(z)|$$

$$\leq |z| + \frac{2\beta\tau(1-\alpha)}{(n+1)[n(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(n+1)}|z|^{n+1}. \tag{26}$$

The result is sharp for

$$f(z) = z - \frac{2\beta\tau(1-\alpha)}{(n+1)[n(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(n+1)} z^{n+1}.$$

Proof. We have

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k,$$

$$|f(z)| \leq |z| + \sum_{k=n+1}^{\infty} a_k |z|^k$$

$$\leq |z| + \frac{2\beta\tau(1-\alpha)}{(n+1)[n(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(n+1)}|z|^{n+1}. \tag{27}$$

Similarly,

$$|f(z)| \geq |z| - \sum_{k=n+1}^{\infty} a_k |z|^k$$

$$\geq |z| - \frac{2\beta\tau(1-\alpha)}{(n+1)[n(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(n+1)}|z|^{n+1}. \quad (28)$$

Combining (27) and (28), we get (26).

5. Extreme Points

In the following theorem, we obtain extreme points for the class $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$.

Theorem 7. *Let* $f_n(z) = z$ *and*

$$f_k(z) = z - \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} z^k,$$

$(k = n+1, n+2, \ldots, n \in \mathbb{N} = \{1, 2, \ldots\})$. *Then* $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=n}^{\infty} \sigma_k f_k(z),$$

where $\sigma_k \geq 0$ *and*

$$\sum_{k=n}^{\infty} \sigma_k = 1.$$

In particular, the extreme points of $T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$ *are the functions* $f_n(z) = z$ *and*

$$f_k(z) = z - \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} z^k, \quad k = n+1, n+2, \ldots.$$

Proof. Let us express *f* as in the above theorem, therefore, we can write

$$f(z) = \sum_{k=n}^{\infty} \sigma_k f_k(z)$$

$$= \sigma_n z + \sum_{k=n+1}^{\infty} \sigma_k \left[z - \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} z^k \right]$$

$$= z\left(\sigma_n + \sum_{k=n+1}^{\infty} \sigma_k\right) - \sum_{k=n+1}^{\infty} \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} \sigma_k z^k$$

$$= z - \sum_{k=n+1}^{\infty} h_k z^k,$$

where

$$h_k = \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} \sigma_k.$$

Therefore, $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$, since

$$\sum_{k=n+1}^{\infty} \frac{h_k k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)} = \sum_{k=n+1}^{\infty} \sigma_k = 1 - \sigma_n < 1.$$

Conversely, suppose that $f \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Then by (17), we may set

$$\sigma_k = \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)} a_k, \quad k \geq n+1$$

and

$$1 - \sum_{k=n+1}^{\infty} \sigma_k = \sigma_n.$$

Then

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$$

$$= z - \sum_{k=n+1}^{\infty} \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)} \sigma_k z^k$$

$$= z - \sum_{k=n+1}^{\infty} \sigma_k (z - f_k(z))$$

$$= z\left(1 - \sum_{k=n+1}^{\infty} \sigma_k\right) + \sum_{k=n+1}^{\infty} \sigma_k f_k(z)$$

$$= \sigma_n z + \sum_{k=n+1}^{\infty} \sigma_k f_k(z) = \sum_{k=n}^{\infty} \sigma_k f_k(z).$$

This completes the proof.

6. Closure Theorem

In the following theorem, we will show that the class $T^{\vartheta, \mu, \nu}(n, \tau, \alpha, \beta)$ is closed under linear combination.

Theorem 8. *Let*

$$f_i(z) = z - \sum_{k=n+1}^{\infty} a_{k,i} z^k \in T^{\vartheta, \mu, \nu}(n, \tau, \alpha, \beta), \quad i \in \{1, 2, ..., t\}$$

and $0 < c_i < 1$ *such that*

$$\sum_{i=1}^{t} c_i = 1.$$

Then the function $F(z)$ *defined*

$$F(z) = \sum_{i=1}^{t} c_i f_i(z)$$

is also in the class $T^{\vartheta, \mu, \nu}(n, \tau, \alpha, \beta)$.

Proof. For every $i \in \{1, 2, ..., t\}$, we obtain

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta, \mu}(k)}{2\beta\tau(1-\alpha)} a_{k,i} \le 1.$$

Since

$$F(z) = \sum_{i=1}^{t} c_i f_i(z) = \sum_{i=1}^{t} c_i \left(z - \sum_{k=n+1}^{\infty} a_{k,i} z^k \right)$$

$$= z - \sum_{k=n+1}^{\infty} \left(\sum_{i=1}^{t} c_i a_{k,i} \right) z^k.$$

Therefore,

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}\left[\sum_{i=1}^{t} c_i a_{k,i}\right]$$

$$= \sum_{i=1}^{t} c_i \left[\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)} a_{k,i}\right]$$

$$\leq \sum_{i=1}^{t} c_i = 1.$$

Hence $F(z) \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$ and therefore the proof is complete.

7. Hadamard Product

Theorem 9. *Let* $f, g \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. *Then*

$$(f * g)(z) = z - \sum_{k=n+1}^{\infty} a_k b_k z^k \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$$

for

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k,$$

where

$$\eta \geq \frac{2\beta^2\tau(1-\alpha)(k-1)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]^2 C_1^{\vartheta,\mu}(k) + 2\beta^2\tau(\alpha-1)[(k-1)(2\tau-1)+2\tau(1-\alpha)]}.$$

Proof. Let $f, g \in T^{\vartheta,\mu,\nu}(n, \tau, \alpha, \beta)$. Then

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)} a_k \leq 1 \tag{29}$$

and

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)} b_k \leq 1. \tag{30}$$

We need to find the smallest number η such that

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\eta+2\eta\tau)+2\eta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\eta\tau(1-\alpha)} a_k b_k \le 1. \tag{31}$$

By Cauchy-Schwarz inequality, we have

$$\sum_{k=n+1}^{\infty} \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)} \sqrt{a_k b_k} \le 1. \tag{32}$$

Thus it is enough to show that

$$\frac{k[(k-1)(1-\eta+2\eta\tau)+2\eta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\eta\tau(1-\alpha)} a_k b_k$$

$$\le \frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)} \sqrt{a_k b_k},$$

that is

$$\sqrt{a_k b_k} \le \frac{[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]\eta}{[(k-1)(1-\eta+2\eta\tau)+2\eta\tau(1-\alpha)]\beta}. \tag{33}$$

From (32),

$$\sqrt{a_k b_k} \le \frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}. \tag{34}$$

Therefore, in view of (33) and (34) it is enough to show that

$$\frac{2\beta\tau(1-\alpha)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}$$

$$\le \frac{[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]\eta}{[(k-1)(1-\eta+2\eta\tau)+2\eta\tau(1-\alpha)]\beta},$$

which simplifies to

$$\eta \ge \frac{2\beta^2\tau(1-\alpha)(k-1)}{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]^2 C_1^{\vartheta,\mu}(k)+2\beta^2\tau(\alpha-1)[(k-1)(2\tau-1)+2\tau(1-\alpha)]}.$$

Theorem 10. *Let* $f,\ g \in T^{\vartheta, \mu, \nu}(n, \tau, \alpha, \beta)$. *Then*

$$h(z) = z - \sum_{k=n+1}^{\infty} (a_k^2 + b_k^2) z^k$$

is in the class $T^{\vartheta, \mu, \nu}(n, \tau, \alpha, \beta)$, *where*

$$\eta \geq \frac{4\beta^2 k\tau(1-\alpha)(k-1)}{[k((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha))]^2 C_1^{\vartheta,\mu}(k) + 4\beta^2\tau(1-\alpha)[k(k-1)-2\tau k(k-1)-2k\tau(1-\alpha)]}.$$

Proof. Let $f,\ g \in T^{\vartheta, \mu, \nu}(n, \tau, \alpha, \beta)$ and so

$$\sum_{k=n+1}^{\infty} \left[\frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}\right]^2 a_k^2 \leq 1 \tag{35}$$

and

$$\sum_{k=n+1}^{\infty} \left[\frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}\right]^2 b_k^2 \leq 1. \tag{36}$$

Adding (35) and (36), we get

$$\sum_{k=n+1}^{\infty} \frac{1}{2}\left[\frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}\right]^2 (a_k^2 + b_k^2) \leq 1. \tag{37}$$

We must show that $h \in T^{\vartheta, \mu, \nu}(n, \tau, \alpha, \beta)$, that is

$$\sum_{k=n+1}^{\infty} \left[\frac{k[(k-1)(1-\eta+2\eta\tau)+2\eta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\eta\tau(1-\alpha)}\right](a_k^2 + b_k^2) \leq 1. \tag{38}$$

In view of (37) and (38) it is enough to show that

$$\frac{k[(k-1)(1-\eta+2\eta\tau)+2\eta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\eta\tau(1-\alpha)} \leq \frac{1}{2}\left[\frac{k[(k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha)]C_1^{\vartheta,\mu}(k)}{2\beta\tau(1-\alpha)}\right]^2,$$

which simplifies to

$$\eta \geq \frac{4\beta^2 k\tau(1-\alpha)(k-1)}{[k((k-1)(1-\beta+2\beta\tau)+2\beta\tau(1-\alpha))]^2 C_1^{\vartheta,\mu}(k)+4\beta^2\tau(1-\alpha)[k(k-1)-2\tau k(k-1)-2k\tau(1-\alpha)]}.$$

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