Prior elicitation for mixed quantile regression with an allometric model

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SUMMARY

In this paper, we introduce methods for conducting Bayesian quantile analysis of an allometric model that includes random effects, with the primary goal of estimating quantiles of the length-weight relationship of fish data. We develop prior elicitation schemes to incorporate historical information into the analysis of current data. We propose Gibbs

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sampler methods to facilitate Bayesian computation and provide detailed implementation schemes. We apply our methods to analyze data of walleye Sander vitreus and white bass Morone chrysops in Lake McConaughy, Nebraska.

Keywords: Allometric model; Elicitation; Mixture representation; Prior distribution; Quantile regression.

1. INTRODUCTION

It is well known that the relationship between the length and weight of fish is given by the following allometric model (Huxley and Teissier, 1936)

$$W = \alpha L^{\beta_1} 10^{\varepsilon}, \tag{1.1}$$

where W is the weight in grams (g), L is the total length in millimeters (mm), α and β_1 are parameters, and ε is the error term. Typically, comparisons between fish of different lengths and from different populations are made using the relative weight (W_r), where $W_r = (W/W_s) \times 100$ and W_s is a standard weight for fish of the same species and length (Wege and Anderson, 1978). Wege and Anderson (1978) developed a standard weight equation using 75th-percentile weights from Carlander (1977). Essentially, the use of 75th percentile weights is based on the premise that the standard weight should represent fish that are in better body condition than the mean body condition (Blackwell *et al.*, 2000). This could be the case when there is no relationship or only a weak relationship between the means of such variables. However, it is useful to look at several quantile-based allometric models in order to discover more useful predictive relationships between the weight and length. Cade *et al.* (2008) showed that at a given total length, the higher to lower quantiles of weight provide sufficient information and a more complete picture of higher to lower body condition for evaluating effects of management actions on individual fish populations.

The estimation of the parameters of an allometric model (1.1) is essential in any analysis of fishery data. In this paper, we use quantile regression to obtain estimates of the quantiles of weight as a function of length. Quantile regression makes minimal assumptions on the error term distribution and thus is able to accommodate non-normal errors, which are common in many applications (Koenker and Bassett, 1978; Koenker, 2005; Brain et al., 2010). In addition, the set of quantiles may give a more complete picture of the relation between the length and weight than mean regression.

We introduce a random effect allometric model, and we employ Bayesian analysis to take account of parameter uncertainty. Moreover, our idea is to estimate the parameters α and β_1 via a Bayesian mixed quantile regression, by incorporating historical data gathered from similar previous studies into the analysis of current data. In fishery studies, large historical databases exist for the length and weight. It is therefore reasonable to incorporate historical data into the analysis of the data, by quantifying it with a suitable prior distribution on the model parameters. There are several methods to incorporate historical data into the analysis of a current study. One of these methods is the power prior proposed by Ibrahim and Chen (2000). This is constructed by raising the likelihood function of the historical data to a power parameter between 0 and 1. Ibrahim and Chen (2000) define the power prior distribution for a set of parameters θ as

$$\pi(\theta|D_0, a_0) \propto \ell(D_0|\theta)^{a_0}\pi_0(\theta),$$

where D_0 denotes the data from the historical study, $\ell(D_0|\theta)$ denotes the likelihood for the historical study, a_0 denotes the power parameter, and the initial prior of θ is $\pi_0(\theta)$. The power parameter represents the proportion of the historical data needed in the current study. We will discuss the selection of a_0 in Section 2.2.

The rest of this paper is organized as follows. Section 2 introduces the linear mixed quantile of an allometric model. Methods for eliciting prior distributions are discussed. Section 3 gives a real data example to illustrate the proposed methodology and to compare our approach with other approaches. The Discussion is in Section 4 and the Appendix contains the details of the Gibbs sampler and full conditional distribution.

2. POSTERIOR INFERENCE

2.1 *Methods*

In an allometric model (1.1), α and β_1 are coefficients determined by fitting the model to observations of length and weight. It is well known that the length-weight relationship of fish varies seasonally and spatially. However, the allometric model has been deemed adequate for many applications (Robertis and Williams, 2008). In clustered data, the allometric model (1.1) can be written as

$$W_{ij} = \alpha L_{ij}^{\beta_1} 10^{\varepsilon_{ij}}, \quad i = 1, ..., N; j = 1, ..., n_i \tag{2.1}$$

where W_{ij} is the jth weight in cluster i, L_{ij} is the jth total length in cluster i and ε_{ij} is the error term. In this paper, we include a random intercept to the allometric model (2.1), in order to allow for heterogeneous intercepts among clusters. If we assume $\alpha = \beta_0 10^{u_i}$, where β_0 is the common constant for all clusters and u_i is the location shift random effect of the ith cluster, then Equation (2.1) can be extended as

$$W_{ij} = \beta_0 L_{ij}^{\beta_1} 10^{u_i \varepsilon_{ij}}, \quad i = 1, ..., N; j = 1, ..., n_i. \tag{2.2}$$

After logarithmic transformation, Equation (2.2) becomes

$$\log_{10} W_{ij} = \log_{10} \beta_0 + \beta_1 \log_{10} L_{ij} + u_i + \varepsilon_{ij}. \tag{2.3}$$

In this model (2.3), we assume that the conditional qauntile of $\log_{10} W_{ij}$, $Q_{\log_{10} W_{ij}|u_i, L_{ij}}$, is given by $\log_{10} \beta_{0p} + \beta_{1p} \log_{10} L_{ij} + u_i$. So, we assume that the pth quantile of model error ε_{ij} is zero. That is,

$$Q_{\log_{10} W_{ij}|u_i,L_{ij}}(p) = \log_{10}\beta_{0p} + \beta_{1p}\log_{10} L_{ij} + u_i. \tag{2.4}$$

For simplicity of notation, we will omit the subscript p in the remainder of the paper. We assume that the u_i are independently and identically distributed (iid) according to a normal distribution with mean zero and variance σ_u^{-2}.

Given observations $\{W_{ij}, L_{ij}, u_i; i = 1, \cdots, N,\ j = 1, \cdots, n_i\}$, the regression coefficients in (2.4) can be estimated consistently as the solution to the following minimization problem (Koenker, 2005)

$$\sum_{i=1}^{N}\sum_{j=1}^{n_i} \rho_p(\log_{10} W_{ij} - \log_{10}\beta_0 - \beta_1 \log_{10} L_{ij} - u_i), \tag{2.5}$$

where $\rho_p(\cdot)$ is the check function defined by

$$\rho_p(t) = \begin{cases} pt, & \text{if } t \geqslant 0, \\ (p-1)t, & \text{if } t < 0. \end{cases}$$

A possible parametric link between the minimization problem in (2.5) and maximum likelihood theory is given by the asymmetric Laplace distribution; see Koenker and Machado (1999), Yu and Moyeed (2001). A random variable $\log_{10} W_{ij}$ is distributed as an asymmetric Laplace distribution with parameters μ_{ij}, τ, and p if the corresponding probability density is given by

$$f(\log_{10} W_{ij}|\mu_{ij}, \tau, p) = \frac{p(1-p)}{\tau}\exp\{-\rho_p(\frac{\log_{10} W_{ij} - \mu_{ij}}{\tau})\}, \tag{2.6}$$

where μ_{ij} is the location parameter, τ is the scale parameter and p determines the quantile level. The parameter p determines the skewness of the distribution and the pth quantile of this distribution is zero. The minimization problem in (2.5) is equivalent to maximizing the likelihood function of $\log_{10} W_{ij}$ by assuming $\log_{10} W_{ij}$ from an asymmetric Laplace distribution with $\mu_{ij} = \log_{10}\beta_0 + \beta_1 \log_{10} L_{ij} + u_i$ and $\tau = 1$. Following this standard asymmetric Laplace error distribution, Yu and Moyeed (2001) implemented Bayesian inference for quantile regression, Yu and Stander (2007) developed a Bayesian estimation procedure for the Tobit Quantile regression, and Geraci and Bottai (2007) account for within-subject correlation by adding a random subject effect to the quantile regression model. In this paper, we assume that the $\log_{10} W_{ij}$, conditional on u_i, for $i = 1, ..., N$ and $j = 1, ..., n_i$, are independently distributed according to an asymmetric Laplace distribution.

Letting $\mathbf{W} = (\log_{10} W_{11}, ..., \log_{10} W_{Nn_N})'$, $\mathbf{L} = (\mathbf{L}_{11}, ..., \mathbf{L}_{Nn_N})'$, $\mathbf{L}'_{ij} = (1, \log_{10} L_{ij})$, $\mathbf{u} = (u_1, ..., u_N)'$ and $\beta = (\log_{10}\beta_0, \beta_1)'$, then the joint density of $(\mathbf{W}, \mathbf{u})$ based on N clusters is given by

$$f(\mathbf{W}, \mathbf{u}|\beta, \tau, \sigma_u^{-2}) = \prod_{i=1}^{N}\prod_{j=1}^{n_i} f(\log_{10} W_{ij}|\beta, u_i, \tau) f(u_i|\sigma_u^{-2}). \tag{2.7}$$

Our interest lies on the likelihood function of $\mathbf{W}$ given β, τ, and σ_u^{-2}. Thus, integrating out the random effect leads to the likelihood

$$f(\mathbf{W}|\beta, \tau, \sigma_u^{-2}) = \int_{R^N} f(\mathbf{W}, \mathbf{u}|\beta, \tau, \sigma_u^{-2}) d\mathbf{u}, \tag{2.8}$$

where R^N denotes the N-dimensional Euclidean space. To proceed with a Bayesian analysis, we need to specify a prior distribution for model parameters β, τ, and σ_u^{-2}. We give details of prior elicitation in the next part.

2.2 *Elicitation of prior distribution*

It is well known that a standard conjugate prior is not available for quantile regression (Yu and Stander, 2007). Thus, Bayesian quantile inference models, including Bayesian parametric, Bayesian semiparametric and Bayesian nonparametric models, either set priors independently of the values of the quantiles, or assume the prior to be the same for modelling different quantiles. In addition, the sample size of fishery data in a given study is typically small compared to the size of the population. Therefore, it is difficult to estimate the parameters of interest precisely and it is more reasonable to set different priors for different quantiles. For example, a 95% quantile regression model should have different parameter values from the median. To improve the precision of estimates, we develop a new prior distribution characterized by a $p-$dependent parameter. Our idea is to set priors based on historical information gathered from similar previous studies.

Although one can use improper priors in Bayesian quantile regression, the inference on current data is expected to be more reliable and sensitive if there exist historical data gathered from similar previous studies. Incorporating historical information into the analysis of new information through a prior distribution provides a natural framework for updating information across studies (Neelon and O'Malley, 2010). In this paper, we use a power prior distribution, because it introduces a power parameter that explicitly controls the amount of weight assigned to the historical data. Such control is important when the sample size of the current data is quite different from the sample size of the historical data or where there is heterogeneity between two studies (Ibrahim and Chen, 2000). In addition, this prior has an attractive property in Bayesian quantile regression as it is dependent upon the quantile level.

Suppose there exists data from one historical fish study. Let W_{0ij} be the jth weight in cluster i, L_{0ij} be the jth total length in cluster i and let u_{0i} be a location shift random effect of the ith cluster for the previous study. Denote by $D_0 = (N_0, \mathbf{W}_0, \mathbf{L}_0)$ the historical data of size N_0 clusters, measuring the same response variable and covariate as the current study, where $\mathbf{W}_0 = (\log_{10} W_{011}, ..., \log_{10} W_{0N_0 n_{0N_0}})'$, $\mathbf{L}_0 = (\mathbf{L}_{011}, ..., \mathbf{L}_{0N_0 n_{0N_0}})'$, $\mathbf{L}'_{0ij} = (1, \log_{10} L_{0ij})$. For the random intercept model, we follow Chen *et al.* (2003) and we define a prior distribution for β taking the form

$$\pi(\beta|D_0,\tau,\sigma_u^{-2},a_0) \propto (\prod_{i=1}^{N_0}\int\prod_{j=1}^{n_{0i}}[f(\log_{10} W_{0ij}|\beta,u_{0i},\tau)]^{a_0} f(u_{0i}|\sigma_u^{-2})du_{0i})\pi_0(\beta), \quad (2.9)$$

where $f(\log_{10} W_{0ij}|\beta,u_{0i},\tau)$ is $f(\log_{10} W_{ij}|\beta,u_i,\tau)$ with $(\log_{10} W_{0ij},u_{0i})$ in place of $(\log_{10} W_{ij},u_i)$, and a_0 is a fixed parameter, $0 \leqslant a_0 \leqslant 1$. The power parameter a_0 represents how data from the previous study is to be used in the current study. There are two special cases for a_0: the first case $a_0 = 0$ corresponds to no incorporation of the data from the previous study relative to the current study. The second case $a_0 = 1$ corresponds to full incorporation of the data from the previous study relative to the current study. Therefore, a_0 controls the influence of the data gathered from previous studies that is similar to the current study. In this paper we assume that the power parameter a_0 is determined by expert opinion about the relevance of the historical data to the current analysis. The prior specification is completed by specifying priors for β, τ and σ_u^{-2}. We specify a 2-dimensional normal distribution with parameter $(\mathbf{b}_0,\mathbf{B}_0)$ for β, an inverse gamma (IΓ) prior with parameter (l_{01},s_{01}) for τ and gamma (Γ) prior with parameter $(\frac{l_{02}}{2},\frac{s_{02}}{2})$ for σ_u^{-2}. Thus, the joint prior distribution takes the form

$$\begin{aligned}\pi(\beta,\tau,\sigma_u^{-2}|D_0,a_0) \propto \prod_{i=1}^{N_0}\int\prod_{j=1}^{n_{0i}}&[f(\log_{10} W_{0ij}|\beta,u_{0i},\tau)]^{a_0} f(u_{0i}|\sigma_u^{-2})du_{0i}\\ &\times \exp\{-\frac{1}{2}(\beta-\mathbf{b}_0)'\mathbf{B}_0^{-1}(\beta-\mathbf{b}_0)\}\\ &\times(\frac{1}{\tau})^{l_{01}+1}\exp\{-\frac{s_{01}}{\tau}\}(\sigma_u^{-2})^{\frac{l_{02}}{2}-1}\exp\{-\sigma_u^{-2}\frac{s_{02}}{2}\}. \quad (2.10)\end{aligned}$$

Therefore, the joint posterior distribution of β, τ, and σ_u^{-2} is given by

$$f(\beta, \tau, \sigma_u^{-2}|D, D_0, a_0) \propto (\prod_{i=1}^{N} \int \prod_{j=1}^{n_i} f(\log_{10} W_{ij}|\beta, u_i, \tau) f(u_i|\sigma_u^{-2}) du_i) \pi(\beta, \tau, \sigma_u^{-2}|D_0, a_0), \quad (2.11)$$

where $D = (N, \mathbf{W}, \mathbf{L})$ represent the current study.

2.3 *Estimation*

The Gibbs sampler by Geman and Geman (1984) is a very popular method for constructing a Markov chain in Bayesian inference, and it is used to generate a sequence of samples from the full conditional distribution. To implement this method, the full conditional posterior distributions of all unknown parameters are needed. In our case, each of these distributions can be obtained by regarding all other parameters in (2.11) as known.

The conditional distribution of the random effect for the current study $f(u_i| \log_{10} W_{ij}, \beta, \tau, \sigma_u^{-2}) \propto f(\log_{10} W_{ij}, u_i|\beta, \tau, \sigma_u^{-2})$ is log-concave in u_i and the conditional distribution of the random effect for the historical data $f(u_{0i}| \log_{10} W_{0ij}, \beta, \tau, \sigma_u^{-2}) \propto f(\log_{10} W_{0ij}, u_{0i}|\beta, \tau, \sigma_u^{-2})$ is log-concave in u_{0i}. Therefore, we can use the Gibbs sampler for Bayesian analysis of the quantile regression model. The details of the Gibbs sampler and full conditional distributions are given in the Appendix. If the historical data are not available, the power prior distribution reduces to the form

$$\begin{aligned}\pi(\beta, \tau, \sigma_u^{-2}|D_0, a_0) &\propto \exp\{-\frac{1}{2}(\beta - \mathbf{b}_0)\mathbf{B}_0^{-1}(\beta - \mathbf{b}_0)\} \\ &\times (\frac{1}{\tau})^{l_{01}+1} \exp\{-\frac{s_{01}}{\tau}\}(\sigma_u^{-2})^{\frac{l_{02}}{2}-1} \exp\{-\sigma_u^{-2}\frac{s_{02}}{2}\}. \quad (2.12)\end{aligned}$$

This case is equivalent to $a_0 = 0$. In the case that there is no historical data available and the current data are non-clustered, then by using the above prior distribution, the Bayesian quantile regression reduces to (Reed and Yu, 2009) and is implemented using the "MCMCquantreg" package in R (R Development Core Team, 2010).

3. ANALYSIS OF THE WALLEYE AND WHITE BASS DATA

We used Bayesian quantile regression to estimate the length-weight relationship of walleye Sander vitreus and white bass Morone chrysops before (1980-1989) and after (1989-2004) in Lake McConaughy, Nebraska. More details about the study area and fish sampling are reported in Porath *et al.* (2003). The data was analyzed by Cade *et al.* (2008). Cade *et al.* (2008) consider fish measured before (1980-1989) as pre-alewife introduction and those measured after (1989-2004) as post-alewife introduction. The authors developed the allometric model by including an indicator variable for the groups in the years before alewife introduction and after. The model is given by

$$Q_{\log_{10} W}(p|L, I) = \log_{10}\beta_0 + \beta_1 \log_{10} L + \beta_2 I + \beta_3 I \log_{10} L, \tag{3.1}$$

where I is an indicator variable taking the value 0 for years (1980-1988) and the value 1 for years (1989-2004). The data are shown graphically in Figures 1 and 2. In comparing weights between before-alewife-introduction and after, we emphasize how weights change with length in Figure 1. In addition we emphasize how weights change over time in Figure 2. The response variable Weight measures the weight (g) of individuals that are classified according to years. Since the data occurs in clusters (years), it is very likely that observations from the same year are statistically correlated and not independent. In this case, it is inappropriate to analyze the data using a linear model. Our primary objective in this study is to estimate the parameters of an allometric model for clustered data (2.2) by incorporating prior historical information. We use post-alewife introduction as current data and pre-alewife introduction as historical data. Then, we incorporate the historical data into the analysis of the current study by quantifying it with a suitable prior distribution on the model parameters.

We compare three models: the Bayesian quantile regression model without historical data, using the “MCMCquantreg” package in R (R Development Core Team, 2010), the standard frequentist quantile regression for Cade’s model (3.1), referred to as “CQR” and obtained using the quantreg package in R with the default rank method to obtain confidence intervals, and our approach with a power prior distribution assuming heterogeneous intercepts among clusters, "GSRE$_{a_0}$". The methods are evaluated based on 95% intervals for each procedure, the mean error (ME) and the root mean square error (RMSE) for each model where

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{N}\sum_{j=1}^{n_i}(\log_{10} W_{ij} - \log_{10}\hat{W}_{ij})^2}{\sum_{i=1}^{N} n_i}} \tag{3.2}$$

We begin our analysis of the walleye and white bass data by separate analysis of current and historical studies ignoring the clusters and using MCMCquantreg. The estimated parameters of β_0 and β_1 vary from approximately $10^{-6.1937}$ and 3.4213 at lower quantiles (p=0.05) to $10^{-5.4150}$ and 3.1930 at higher quantiles (p=0.95), respectively, for walleye captured in 1989-2004 (current data), whereas the estimated parameters vary from approximately $10^{-5.9395}$ and 3.3154 at lower quantiles to $10^{-5.4377}$ and 3.1831 at higher quantiles, respectively, for walleye captured in 1980-1988 (historical data). For white bass data, the estimated parameters of β_0 and β_1 vary from approximately $10^{-6.2614}$ and 3.4883 at lower quantiles to $10^{-4.6035}$ and 2.9564 at higher quantiles, respectively, for white bass captured in 1989-2004 (current data), and from approximately $10^{-5.6033}$ and 3.2555 at lower quantiles to $10^{-4.8744}$ and 3.0316 at higher quantiles, respectively, for white bass captured in 1980-1988 (historical data).

Next, we incorporate the historical data into the analysis of the current study using the allometric model (3.1). We estimate quantiles by increments of 0.01 from 0.05 to 0.95. We can see from Figure 3 that the bands of parameter estimates and their confidence intervals become wider for quantiles greater than 0.80 for the walleye data. For white bass data, the bands of parameter estimates and their confidence intervals narrow in estimates from lower to higher quantiles, indicating heterogeneity of the data.

There are substantial differences in the parameter estimates between the current and historical data and the sample sizes of each are quite different. Furthermore, there is evidence of heterogeneity in the data, and therefore the full incorporation of the historical data into the analysis of the current study is considered inappropriate. Thus, we use the power prior approach, where we assume a range of values for a_0 ($a_0 = 0.25$ and 0.50). Moreover, we treat the data as clustered data. We assume no prior knowledge and use independent $N(0, 10^3)$ priors on all regression parameters, $l_{01} = s_{01} = 0.01$ and $l_{02} = s_{02} = 1$. We run our Gibbs sampler for 11000 iterations with an initial burn-in of 1000 iterations and we estimate quantiles by increments of 0.01 from 0.05 to 0.95.

Tables 1 and 2 summarise the 95% intervals for the three approaches at 5 different quantiles, namely 0.05, 0.25, 0.50, 0.75 and 0.95. Clearly, the 95% intervals are generally much wider than our intervals ($GSRE_{a_0}$). For example in the walleye data, at $p = 0.95$ and a_0=0.50, the interval width of our approach for $\log_{10} \beta_0$ and β_1 is 0.024 and 0.049, respectively, compared to an interval width of 0.163 and 0.068 for CQR, respectively, and 1.414 and 0.636 for the MCMCquantreg method, respectively. Similarly, in the white bass data, at $p = 0.95$ and a_0=0.50, the interval width of our approach for $\log_{10} \beta_0$ and β_1 is 0.089 and 0.034, respectively, compared to an interval width of 0.164 and 0.066 for CQR, respectively, and 5.849 and 2.477 for the MCMCquantreg method, respectively.

Tables 1 and 2 summarise the comparison of results also in terms of ME and RMSE for each model. Clearly, the mean error of the three approaches are more or less the same. However, in general, the mean error in our model is smaller than the mean error results obtained from MCMCquantreg and CQR . In addition, the root mean squared error comparison shows that our model is more efficient than MCMCquantreg and CQR.

4. DISCUSSION

In this paper we have introduced Bayesian quantile regression methods for an allometric model, with the primary goal of estimating models that include random effects. We have elicited a prior distribution from the historical data. The prior distribution depends on the quantile level. Thus, we have different priors for different quantiles. The prior distributions can be used either with no prior data or with complete prior data. Full conditional distributions have been outlined for unknown parameters and a Gibbs sampler is derived.

The results show that our method generally performs better than the others across the data in terms of the 95% intervals, ME and RMSE. We believe that our proposed method will allow researchers to obtain more precise estimates of the length-weight relationship by incorporating historical fish data into the analysis of current fish studies.

5. Appendix A

As shown in Reed and Yu (2009) and Kozumi and Kobayashi (2009), any variable with an asymmetric Laplace distribution can be represented as a scale mixture of normals:

$$\varepsilon =^d \frac{1-2p}{p(1-p)}\tau z + \sqrt{\frac{2z}{p(1-p)}}\tau\xi. \tag{5.1}$$

The random variables $z > 0$ and ξ are independent and have standard exponential distribution and standard normal distribution, respectively.

Let

$$\theta = \frac{1-2p}{p(1-p)} \qquad \text{and} \qquad \phi^2 = \frac{2}{p(1-p)}.$$

We assume that $\log_{10} W_{ij}$, conditional on u_i and z_{ij}, for $i = 1, ..., N$ and $j = 1, ..., n_i$, are independently distributed according to a normal distribution with mean $\mathbf{L}'_{ij}\beta + u_i + \tau\theta z_{ij}$ and variance $\tau^2\phi^2 z_{ij}$. From now on, it is more convenient with the Gibbs sampler to work with $v_{ij} = \tau z_{ij}$ to avoid the scale parameter τ in the conditional mean of $\log_{10} W_{ij}$. Thus, the conditional densities of the response $\log_{10} W_{ij}$ and $\log_{10} W_{0ij}$ are given, respectively, by

$$f(\log_{10} W_{ij}|\beta, u_i, \tau, v_{ij}) \propto (\tau v_{ij})^{-\frac{1}{2}} \exp\{-\frac{(\log_{10} W_{ij} - \mathbf{L}'_{ij}\beta - u_i - \theta v_{ij})^2}{2\tau\phi^2 v_{ij}}\},$$

$$f(\log_{10} W_{0ij}|\beta, u_{0i}, \tau, v_{0ij}) \propto (\tau v_{0ij})^{-\frac{1}{2}} \exp\{-\frac{(\log_{10} W_{0ij} - \mathbf{L}'_{0ij}\beta - u_{0i} - \theta v_{0ij})^2}{2\tau\phi^2 v_{0ij}}\},$$

where v_{0ij} represents the exponential variable. The posterior distribution can then be calculated using Bayes theorem

$$\begin{aligned} f(\beta, \tau, \sigma_u^{-2}, \mathbf{u}, \mathbf{u}_0, \mathbf{v}, \mathbf{v}_0|D, D_0, a_0) \propto \prod_{i=1}^{N}\prod_{j=1}^{n_i} f(\log_{10} W_{ij}|\beta, u_i, \tau, v_{ij}) f(u_i|\sigma_u^{-2})\pi(v_{ij}|\tau) \\ \times \prod_{i=1}^{N_0}\prod_{j=1}^{n_{oi}} [f(\log_{10} W_{0ij}|\beta, u_{0i}, \tau, v_{0ij})]^{a_0} f(u_{0i}|\sigma_u^{-2})\pi(v_{0ij}|\tau) \\ \times\pi(\beta)\pi(\tau)\pi(\sigma_u^{-2}), \end{aligned} \tag{5.2}$$

where $\mathbf{v} = (v_{11}, ..., v_{N_{n_N}})'$ and $\mathbf{v}_0 = (v_{011}, ..., v_{0N_{0n_{0N_0}}})'$. A little algebra shows that

$$B_p^{-1} = \sum_{i=1}^{N}\sum_{j=1}^{n_i} \frac{\mathbf{L}_{ij}\mathbf{L}'_{ij}}{\tau\phi^2 v_{ij}} + a_0 \sum_{i=1}^{N_0}\sum_{j=1}^{n_{0i}} \frac{\mathbf{L}_{0ij}\mathbf{L}'_{0ij}}{\tau\phi^2 v_{0ij}} + \mathbf{B}_0^{-1},$$

and

$$\hat{\beta} = B_p(\sum_{i=1}^{N}\sum_{j=1}^{n_i} \frac{\mathbf{L}_{ij}(\log_{10} W_{ij} - u_i - \theta v_{ij})}{\tau\phi^2 v_{ij}} + a_0 \sum_{i=1}^{N_0}\sum_{j=1}^{n_{0i}} \frac{\mathbf{L}_{0ij}(\log_{10} W_{0ij} - u_{0i} - \theta v_{0ij})}{\tau\phi^2 v_{0ij}} + \mathbf{B}_0^{-1}\mathbf{b}_0).$$

Then, given $\tau, \sigma_u^{-2}, \mathbf{u}, \mathbf{u}_0, \mathbf{v}, \mathbf{v}_0$ and a_0, we have $\beta|\tau, \sigma_u^{-2}, \mathbf{u}, \mathbf{u}_0, \mathbf{v}, \mathbf{v}_0, a_0, D, D_0 \sim$

$N_2(\hat{\beta}, B_p)$, where N_2 denotes a 2-dimensional normal distribution. From (5.2), we have that the conditional posterior distribution of v_{ij} given β, τ and u_i is the kernel of a generalized inverse Gaussian distribution; that is, $v_{ij}|\beta, \tau, u_i, D \sim GIG(1/2, a_1, b_1)$, where $a_1 = ((\log_{10} W_{ij} - \mathbf{L}'_{ij}\beta - u_i)^2/\tau\phi^2)^{1/2}$, $b_1 = (2/\tau + \theta^2/\tau\phi^2)^{1/2}$ and the density of the $GIG(\frac{1}{2}, a_1, b_1)$ is

$$\pi(v_{ij}|\frac{1}{2}, a_1, b_1) \propto v_{ij}^{\frac{-1}{2}} \exp\{-\frac{1}{2}[(\frac{(\log_{10} W_{ij} - \mathbf{L}'_{ij}\beta - u_i)^2}{\tau\phi^2})v_{ij}^{-1} + (\frac{2}{\tau} + \frac{\theta^2}{\tau\phi^2})v_{ij}]\}.$$

From (5.2) we can deduce that the full conditional posterior distribution of v_{0ij} given β, τ and u_{0i} is also the kernel of a generalized inverse Gaussian distribution, that is $v_{0ij}|\beta, \tau, u_{0i}, a_0, D, D_0 \sim GIG((2 - a_0)/2, a_2, b_2)$, where $a_2 = (a_0(\log_{10} W_{0ij} - \mathbf{L}'_{0ij}\beta$ $-u_{0i})^2/\tau\phi^2)^{1/2}$, and $b_2 = (2/\tau + a_0\theta^2/\tau\phi^2)^{1/2}$. The conditional distribution of τ given $\beta, \mathbf{u}$, and $\mathbf{u}_0$ is a Gamma distribution, that is $\tau|\beta, \mathbf{u}, \mathbf{u}_0, a_0, D, D_0 \sim G(a_3, b_3)$, where

$$a_3 = l_{01} + \sum_{i=1}^{N} n_i + a_0 \sum_{i=1}^{N_0} n_{0i}$$

and

$$b_3 = s_{01} + \sum_{i=1}^{N}\sum_{j=1}^{n_i} \rho_p(\log_{10} W_{ij} - \mathbf{L}'_{ij}\beta - u_i) + a_0 \sum_{i=1}^{N_0}\sum_{j=1}^{n_{0i}} \rho_p(\log_{10} W_{0ij} - \mathbf{L}'_{0ij}\beta - u_{0i}).$$

The conditional posterior distribution of σ_u^{-2} given $\mathbf{u}$, $\mathbf{u}_0$ and a_0 is a Gamma distribution, that is

$$\sigma_u^{-2}|\mathbf{u}, \mathbf{u}_0, a_0, D, D_0 \sim G(\frac{1}{2}(l_{02} + N + a_0 N_0), \frac{1}{2}(s_{02} + \sum_{i=1}^{N} u_i^2 + a_0 \sum_{i=1}^{N_0} u_{0i}^2)).$$

The conditional posterior distribution of u_i given $\beta, \tau, \sigma_u^{-2}$ and v_{ij} is normal with mean

$$\frac{\sum_{j=1}^{n_i} v_{ij}}{n_i + \tau\phi^2\sigma_u^2 \sum_{j=1}^{n_i} v_{ij}} \sum_{j=1}^{n_i} \frac{(\log_{10} W_{ij} - L'_{ij}\beta - \theta v_{ij})}{v_{ij}},$$

and variance

$$\frac{\tau\phi^2 \sum_{j=1}^{n_i} v_{ij}}{n_i + \tau\phi^2\sigma_u^2 \sum_{j=1}^{n_i} v_{ij}}.$$

Finally, the conditional posterior distribution of u_{0i} given $\beta, \tau, \sigma_u^{-2}$ and v_{0ij} is normal with mean

$$\frac{\sum_{j=1}^{n_{0i}} v_{0ij}}{a_0 n_{0i} + \tau\phi^2\sigma_u^2 \sum_{j=1}^{n_{0i}} v_{0ij}} \sum_{j=1}^{n_{0i}} a_0\left(\frac{(\log_{10} W_{0ij} - L'_{0ij}\beta - \theta v_{0ij})}{v_{0ij}}\right),$$

and variance

$$\frac{\tau\phi^2 \sum_{j=1}^{n_{0i}} v_{0ij}}{a_0 n_{0i} + \tau\phi^2\sigma_u^2 \sum_{j=1}^{n_{0i}} v_{0ij}}.$$

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Table 1. 95% intervals, ME and RMSE for the walleye data. The power prior approach assuming heterogeneous intercepts among years ($GSRE_{a_0}$) is compared with two other approaches: the MCMCquantreg method for the current data and the frequentist quantile regression method of Cade's model(CQR).

Model	p	β_0	β_1	ME	RMSE
MCMCquantreg	0.95	(-6.121, -4.707)	(2.821, 3.457)	0.068	0.083
CQR	0.95	(-5.491, -5.328)	(3.135, 3.203)	0.026	0.076
$GSRE_{a_0=0.25}$	0.95	(-5.477, -5.414)	(3.142, 3.197)	0.017	0.055
$GSRE_{a_0=0.50}$	0.95	(-5.442, -5.418)	(3.139, 3.188)	0.018	0.050
MCMCquantreg	0.75	(-6.060, -5.325)	(3.133, 3.411)	0.025	0.052
CQR	0.75	(-5.642, -5.539)	(3.203, 3.253)	-0.053	0.096
$GSRE_{a_0=0.25}$	0.75	(-5.608, -5.575)	(3.205, 3.246)	0.015	0.019
$GSRE_{a_0=0.50}$	0.75	(-5.609, -5.571)	(3.211, 3.235)	0.013	0.021
MCMCquantreg	0.50	(-6.155, -5.490)	(3.183, 3.437)	0.008	0.057
CQR	0.50	(-5.818, -5.634)	(3.235, 3.294)	-0.073	0.103
$GSRE_{a_0=0.25}$	0.50	(-5.757, -5.655)	(3.242, 3.275)	0.017	0.039
$GSRE_{a_0=0.50}$	0.50	(-5.766, -5.661)	(3.248, 3.278)	0.013	0.037
MCMCquantreg	0.25	(-6.331, -5.441)	(3.193, 3.491)	-0.046	0.063
CQR	0.25	(-5.841, -5.737)	(3.257, 3.300)	-0.097	0.121
$GSRE_{a_0=0.25}$	0.25	(-5.821, -5.776)	(3.261, 3.300)	-0.010	0.029
$GSRE_{a_0=0.50}$	0.25	(-5.824, -5.779)	(3.264, 3.302)	-0.014	0.023
MCMCquantreg	0.05	(-6.881, -5.545)	(3.175, 3.667)	-0.067	0.137
CQR	0.05	(-5.992, -5.915)	(3.307, 3.334)	-0.194	0.215
$GSRE_{a_0=0.25}$	0.05	(-5.971, -5.947)	(3.317, 3.324)	-0.039	0.037
$GSRE_{a_0=0.50}$	0.05	(-5.971, -5.944)	(3.318, 3.324)	-0.043	0.032

Table 2. 95% intervals, ME and RMSE for the white bass data. The power prior approach assuming heterogeneous intercepts among years (GSRE_{a_0}) is compared with two other approaches: the MCMCquantreg method for the current data and the frequentist quantile regression method of Cade's model(CQR).

Model	p	β_0	β_1	ME	RMSE
MCMCquantreg	0.95	(-7.407, -1.558)	(1.729, 4.176)	0.098	0.124
CQR	0.95	(-5.078, -4.914)	(3.042, 3.108)	0.231	0.258
$\text{GSRE}_{a_0=0.25}$	0.95	(-5.081, -4.984)	(3.058, 3.109)	0.042	0.071
$\text{GSRE}_{a_0=0.50}$	0.95	(-5.077, -4.988)	(3.073, 3.107)	0.031	0.059
MCMCquantreg	0.75	(-6.232, -3.936)	(2.636, 3.580)	0.049	0.087
CQR	0.75	(-5.356, -5.177)	(3.128, 3.202)	0.101	0.142
$\text{GSRE}_{a_0=0.25}$	0.75	(-5.308, -5.247)	(3.141, 3.172)	0.021	0.034
$\text{GSRE}_{a_0=0.50}$	0.75	(-5.301, -5.249)	(3.145, 3.183)	0.025	0.031
MCMCquantreg	0.50	(-6.511, -4.355)	(2.786, 3.671)	0.010	0.084
CQR	0.50	(-5.525, -5.197)	(3.114, 3.249)	-0.005	0.117
$\text{GSRE}_{a_0=0.25}$	0.50	(-5.429, -5.316)	(3.125, 3.198)	0.001	0.022
$\text{GSRE}_{a_0=0.50}$	0.50	(-5.411, -5.302)	(3.128, 3.207)	0.003	0.012
MCMCquantreg	0.25	(-7.108, -4.446)	(2.790, 3.890)	-0.047	0.087
CQR	0.25	(-5.621, -5.187)	(3.109, 3.274)	-0.286	0.320
$\text{GSRE}_{a_0=0.25}$	0.25	(-5.558, -5.271)	(3.172, 3.233)	-0.023	0.041
$\text{GSRE}_{a_0=0.50}$	0.25	(-5.516, -5.316)	(3.178, 3.228)	-0.029	0.047
MCMCquantreg	0.05	(-9.887, -3.331)	(2.240, 4.952)	-0.127	0.150
CQR	0.05	(-5.897, -5.146)	(3.072, 3.382)	-0.765	0.815
$\text{GSRE}_{a_0=0.25}$	0.05	(-5.522, -5.228)	(3.061, 3.272)	-0.039	0.064
$\text{GSRE}_{a_0=0.50}$	0.05	(-5.531, -5.297)	(3.078, 3.229)	-0.034	0.061

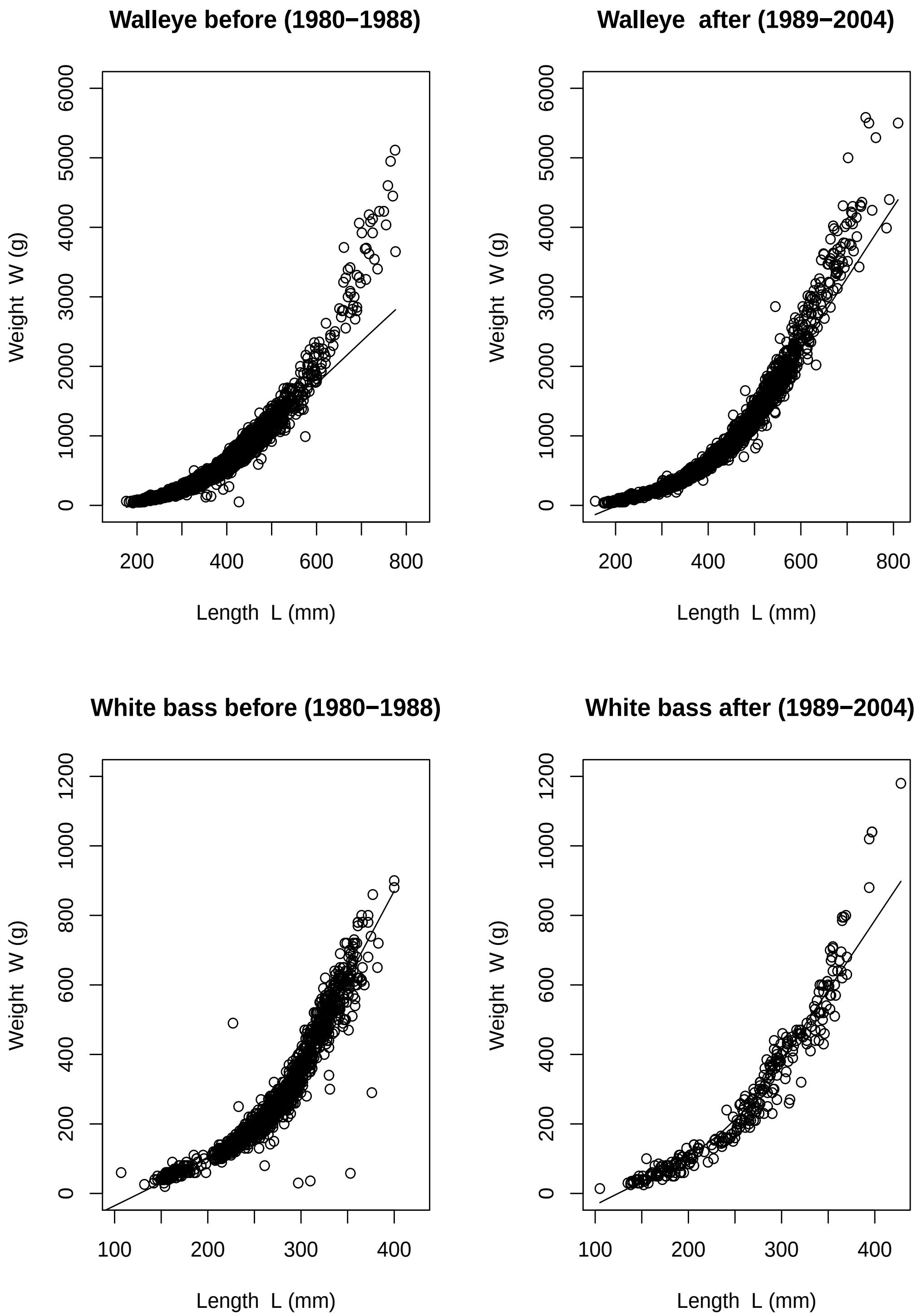

Fig. 1. Scatter plot of the walleye Sander vitreus and white bass Morone chrysops data before (1980-1988) and after (1989-2004). A smooth fitted line shows the change of the weight in grams against the total length in mm.

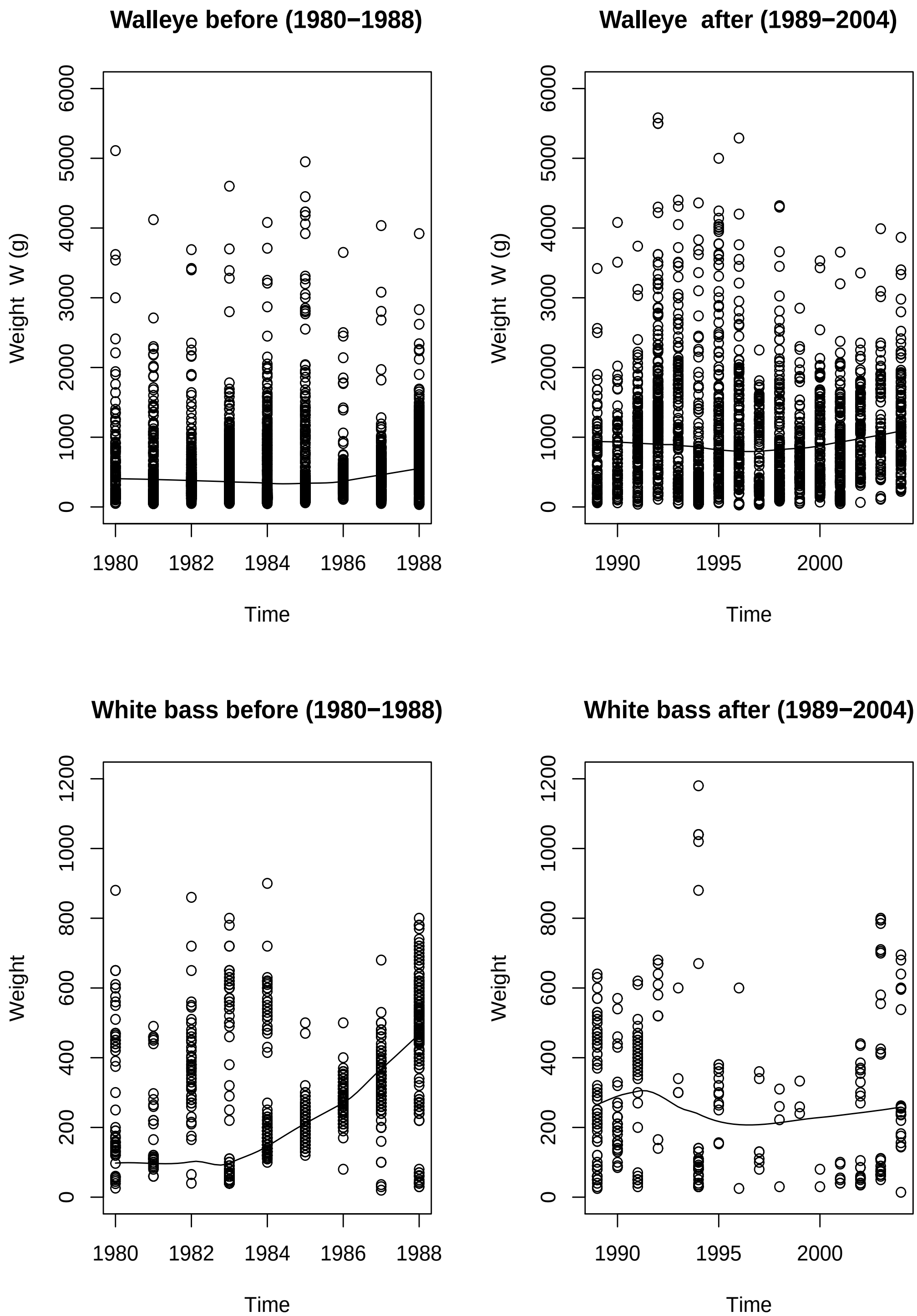

Fig. 2. Scatter plot of the walleye Sander vitreus and white bass Morone chrysops data before (1980-1988) and after (1989-2004). A smooth fitted line shows the change of the weight in grams over time.

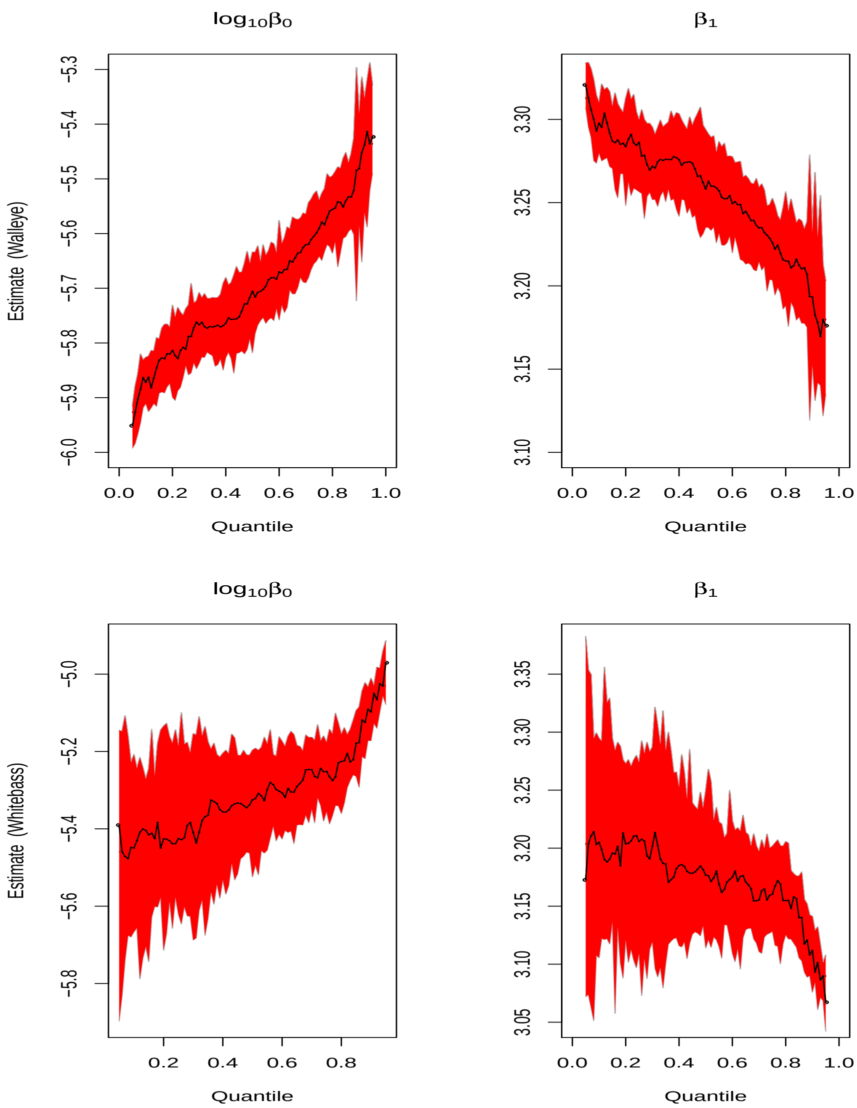

Fig. 3. 95% confidence intervals of the quantile regression coefficients for the CQR model.