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# ON A CERTAIN CLASS OF MULTIVALENT FUNCTIONS DEFINED BY HADAMARD PRODUCT

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## ABSTRACT

*The object of this paper to study the class $WH(n, p, \alpha, \lambda, \theta, q)$ of multivalent functions defined by Hadamard product in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. We obtain some geometric properties for this class, like, coefficient estimate, closure theorem, extreme points, distortion theorem and modified Hadamard products.*

**Keywords:** Multivalent Function, Hadamard Product, Closure Theorem, Distortion Theorem, Extreme Points, Modified Hadamard Products.

**2000 Mathematics Subject Classification:** 30C45, 30C50

## 1. INTRODUCTION

Let $W(p, n)$ denote the class of functions of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \ (a_k \geq 0\,; k \geq n+p; p, n \in \mathbb{N} = \{1,2,\dots\}), \qquad (1.1)$$

which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.
If $f \in W(p, n)$ is given by (1.1) and $g \in W(p, n)$ given by

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \,, \qquad (b_k \geq 0\,)$$

then the Hadamard product (or convolution) $f * g$ of $f$ and $g$ is defined by

$$(f * g)(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z). \qquad (1.2)$$

A function $f \in W(p, n)$ is said to be multivalent starlike of order $\alpha$ $(0 \leq \alpha < p)$ if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U;\ 0 \leq \alpha < p; p \in \mathbb{N}), \qquad (1.3)$$

and is said to be multivalent convex of order $\alpha$ $(0 \le \alpha < p)$ if it satisfies the condition:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U;\ 0 \le \alpha < p; p \in \mathbb{N}). \tag{1.4}$$

Denote by $S_n^*(p,\alpha)$ and $C_n(p,\alpha)$ the classes of multivalent starlike and multivalent convex functions of order $\alpha$, respectively, which were introduced and studied by Owa (1992). It is known that (see (Goodman, 1983) and (Owa, 1992))

$$f \in C_n(p,\alpha) \text{ if and only if } \frac{zf'(z)}{p} \in S_n^*(p,\alpha). \tag{1.5}$$

The classes $S_1^*(p,\alpha) = S^*(p,\alpha)$ and $C_1(p,\alpha) = C(p,\alpha)$ were studied by Owa (1985).

**Definition 1.1.** Let $f$ be given by (1.1), is said to be in the class $WH(n,p,\alpha,\lambda,\theta,q)$ if and only if satisfies the inequality:

$$Re\left\{\frac{z\big((f*g)(z)\big)^{(1+q)} + \lambda z^2\big((f*g)(z)\big)^{(2+q)}}{(1-\lambda)\big((f*g)(z)\big)^{(q)} + \lambda z\big((f*g)(z)\big)^{(1+q)}}\right\}$$

$$\ge \theta\left|\frac{z\big((f*g)(z)\big)^{(1+q)} + \lambda z^2\big((f*g)(z)\big)^{(2+q)}}{(1-\lambda)\big((f*g)(z)\big)^{(q)} + \lambda z\big((f*g)(z)\big)^{(1+q)}} - 1\right| + \alpha, \tag{1.6}$$

where $0 \le \alpha < p-q, p > q, n \in \mathbb{N}, q \in \mathbb{N}_0 = \{0,1,2,\dots\}, 0 \le \lambda \le 1, \theta \ge 0$ and for each $f \in W(p,n)$, we have

$$f^{(q)}(z) = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \delta(k,q)a_k z^{k-q}, \tag{1.7}$$

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} 1 & (j=0) \\ i(i-1)\dots(i-j+1) & (j \ne 0) \end{cases}. \tag{1.8}$$

We note that by specializing the parameters $\lambda, \theta, q, n, p,$ we obtain the following different subclasses as studied by various authors:

(1) If $\theta = 0, \lambda = 0,$ the family $WH(n,p,\alpha,\lambda,\theta,q)$ reduces to the class $TS_g^*(p,q,n,\alpha)$ which was studied by Aouf and Mostafa (2012).

(2) If $\theta = 0, \lambda = 1,$ the family $WH(n,p,\alpha,\lambda,\theta,q)$ reduces to the class $TC_g(p,q,n,\alpha)$ which was studied by Aouf and Mostafa (2012).

(3) If $n = p = 1$ and $q = 0,$ the family $WH(n,p,\alpha,\lambda,\theta,q)$ reduces to the class $WR(\lambda,\theta,\alpha)$ which was studied by Atshan and Buti (2011).

(4) If $\theta = 0, \lambda = 0, q = 0$ and replace $n+p$ by $m$, we have $WH(n,p,\alpha,0,0,0) = WH(p,m,\alpha)$ which was studied by Ali *et al.* (2006).

(5) If $\theta = 0, \lambda = 0, q = 0$ and $b_k = 1$ $(k \ge n+p)$, we have

$$WH(n,p,\alpha,0,0,0) = \begin{cases} T_n^*(p,\alpha) & \text{(Owa (1992))} \\ T_\alpha(p,\alpha) & \text{(Yamakawa (1992))} \end{cases}.$$

(6) $\theta = 0, \lambda = 1, q = 0$ and $b_k = 1$ $(k \ge n+p)$, we have

$$WH(n,p,\alpha,1,0,0) = \begin{cases} C_n(p,\alpha) & \text{(Owa (1992))} \\ C_\alpha(p,\alpha) & \text{(Yamakawa (1992))} \end{cases}.$$

(7) If $k = m, \theta = k, \alpha = \beta$ and $b_k = 1$, the family $WH(n, p, \alpha, \lambda, \theta, q)$ reduces to the class $k - UCV_p^n(\lambda, \beta, q)$ which was studied by Aqlan (2004).

**Lemma 1 (**Aqlan (2004) **).** Let $w = u + iv$. Then $Re(w) \geq \sigma$ if and only if $|w - (1 + \sigma)| \leq |w + (1 - \sigma)|$.

**Lemma 2 (**Aqlan (2004) **).** Let $w = u + iv$ and $\sigma, \eta$ are real numbers. Then $Re(w) \geq \sigma|w - 1| + \eta$ if and only if $Re\{w(1 + \sigma e^{i\phi}) - \sigma e^{i\phi}\} > \eta$.

## 2. COEFFICIENT ESTIMATE

**Theorem 1.** Let the function $f$ be in the form (1.1). Then $f$ is in the class $WH(n, p, \alpha, \lambda, \theta, q)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{k!\left(1 + \lambda(k - q - 1)\right)[(k - q)(1 + \theta) - (\theta + \alpha)]}{(k - q)!} a_k b_k$$

$$\leq \frac{p!\left(1 + \lambda(p - q - 1)\right)(p - q - \alpha)}{(p - q)!}, \tag{2.1}$$

where $p, n \in \mathbb{N}, q \in \mathbb{N}_0, k \geq n + p, \theta \geq 0, 0 \leq \alpha < p - q, p > q,$ and $0 \leq \lambda \leq 1$.

The result is sharp for the function $f$ given by

$$f(z) = z^p - \frac{p!(n+p-q)!\left(1+\lambda(p-q-1)\right)(p-q-\alpha)}{(n+p)!(p-q)!\left(1+\lambda(n+p-q-1)\right)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}} z^{n+p}, \tag{2.2}$$

$(p, n \in \mathbb{N};\ p > q;\ q \in \mathbb{N}_0;\ z \in U)$.

**Proof:** Let $f \in WH(n, p, \alpha, \lambda, \theta, q)$. Then $f$ satisfies the inequality (1.6) which is equivalent to

$$Re\left\{\frac{z\left((f * g)(z)\right)^{(1+q)} + \lambda z^2\left((f * g)(z)\right)^{(2+q)}}{(1 - \lambda)\left((f * g)(z)\right)^{(q)} + \lambda z\left((f * g)(z)\right)^{(1+q)}}\left(1 + \theta e^{i\phi}\right) - \theta e^{i\phi}\right\}$$

$\geq \alpha$, (by using Lemma 2)

$(0 \leq \alpha < p - q;\ p > q;\ \theta \geq 0;\ 0 \leq \lambda \leq 1;\ p \in \mathbb{N};\ q \in \mathbb{N}_0$ and $-\pi < \phi \leq \pi$. Or

$$Re\left\{\frac{\left[z\left((f * g)(z)\right)^{(1+q)} + \lambda z^2\left((f * g)(z)\right)^{(2+q)}\right]\left(1 + \theta e^{i\phi}\right)}{(1 - \lambda)\left((f * g)(z)\right)^{(q)} + \lambda z\left((f * g)(z)\right)^{(1+q)}}\right.$$

$$\left. - \frac{\theta e^{i\phi}\left[(1 - \lambda)\left((f * g)(z)\right)^{(q)} + \lambda z\left((f * g)(z)\right)^{(1+q)}\right]}{(1 - \lambda)\left((f * g)(z)\right)^{(q)} + \lambda z\left((f * g)(z)\right)^{(1+q)}}\right\} \geq \alpha. \tag{2.3}$$

Let $g(z) = \left[z\left((f * g)(z)\right)^{(1+q)} + \lambda z^2\left((f * g)(z)\right)^{(2+q)}\right]\left(1 + \theta e^{i\phi}\right)$

$$-\theta e^{i\phi}\left[(1 - \lambda)\left((f * g)(z)\right)^{(q)} + \lambda z\left((f * g)(z)\right)^{(1+q)}\right]$$

$h(z) = (1 - \lambda)\left((f * g)(z)\right)^{(q)} + \lambda z\left((f * g)(z)\right)^{(1+q)}$.

Then (2.3) is equivalent to

$|g(z) + (1 - \alpha)h(z)| \geq |g(z) - (1 + \alpha)h(z)|$ for $0 \leq \alpha < p - q$. (by using Lemma 1)

Now

$$|g(z)+(1-\alpha)h(z)| = \Bigg|\Bigg[\frac{p!}{(p-q-1)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{k!\,a_k b_k}{(k-q-1)!}z^{k-q} + \frac{\lambda p!}{(p-q-2)!}z^{p-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{\lambda k!\,a_k b_k}{(k-q-2)!}z^{k-q}\Bigg]\left(1+\theta e^{i\phi}\right) - \theta e^{i\phi}\Bigg[\frac{(1-\lambda)p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{(1-\lambda)k!\,a_k b_k}{(k-q)!}z^{k-q}$$

$$+\frac{\lambda p!}{(p-q-1)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{\lambda k!\,a_k b_k}{(k-q-1)!}z^{k-q}\Bigg]$$

$$+(1-\alpha)\Bigg[\frac{(1-\lambda)p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{(1-\lambda)k!\,a_k b_k}{(k-q)!}z^{k-q}$$

$$+\frac{\lambda p!}{(p-q-1)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{\lambda k!\,a_k b_k}{(k-q-1)!}z^{k-q}\Bigg]\Bigg|$$

$$= \Bigg|\frac{p!}{(p-q)!}\big(1+\lambda(p-q-1)\big)(p-q+1-\alpha)z^{p-q}$$

$$+\frac{\theta e^{i\phi}p!}{(p-q)!}\big[(p-q)\big(1+\lambda(p-q-2)\big)-1+\lambda\big]z^{p-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}\big(1+\lambda(k-q-1)\big)(k-q+1-\alpha)a_k b_k z^{k-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{\theta e^{i\phi}k!}{(k-q)!}\big[(k-q)\big(1+\lambda(k-q-2)\big)-1+\lambda\big]a_k b_k z^{k-q}\Bigg|$$

$$\geq \frac{p!}{(p-q)!}\big(1+\lambda(p-q-1)\big)(p-q+1-\alpha)|z|^{p-q}$$

$$+\frac{\theta p!}{(p-q)!}\big[(p-q)\big(1+\lambda(p-q-2)\big)-1+\lambda\big]|z|^{p-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}\big(1+\lambda(k-q-1)\big)(k-q+1-\alpha)a_k b_k|z|^{k-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{\theta k!}{(k-q)!}\big[(k-q)\big(1+\lambda(k-q-2)\big)-1+\lambda\big]a_k b_k|z|^{k-q}.$$

Similarly,

$$|g(z)-(1+\alpha)h(z)| = \Bigg|\frac{p!}{(p-q)!}\big(1+\lambda(p-q-1)\big)(p-q-1-\alpha)z^{p-q}$$

$$+\frac{\theta e^{i\phi}p!}{(p-q)!}\big[(p-q)\big(1+\lambda(p-q-2)\big)-1+\lambda\big]z^{p-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}\big(1+\lambda(k-q-1)\big)(k-q-1-\alpha)a_k b_k z^{k-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{\theta e^{i\phi}k!}{(k-q)!}\big[(k-q)\big(1+\lambda(k-q-2)\big)-1+\lambda\big]a_k b_k z^{k-q}\Bigg|$$

$$\leq\frac{p!}{(p-q)!}\big(1+\lambda(p-q-1)\big)(p-q-1-\alpha)|z|^{p-q}$$

$$+\frac{\theta p!}{(p-q)!}\big[(p-q)\big(1+\lambda(p-q-2)\big)-1+\lambda\big]|z|^{p-q}$$

$$+\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}\big(1+\lambda(k-q-1)\big)(k-q-1-\alpha)a_k b_k|z|^{k-q}$$

$$+\sum_{k=n+p}^{\infty}\frac{\theta k!}{(k-q)!}\big[(k-q)\big(1+\lambda(k-q-2)\big)-1+\lambda\big]a_k b_k|z|^{k-q}.$$

Therefore

$$|g(z)+(1-\alpha)h(z)|-|g(z)-(1+\alpha)h(z)|\geq\frac{2p!}{(p-q)!}\big(1+\lambda(p-q-1)\big)(p-q-\alpha)$$

$$-\sum_{k=n+p}^{\infty}\frac{2k!}{(k-q)!}\Big[\big(1+\lambda(k-q-1)\big)(k-q-\alpha)$$

$$+\theta\Big((k-q)\big(1+\lambda(k-q-2)\big)-1+\lambda\Big)\Big]a_k b_k$$

$$\geq 0.$$

Hence

$$\sum_{k=n+p}^{\infty}\frac{k!\big(1+\lambda(k-q-1)\big)}{(k-q)!}[(k-q)(1+\theta)-(\theta+\alpha)]a_k b_k$$

$$\leq\frac{p!\,(p-q-\alpha)}{(p-q)!}\big(1+\lambda(p-q-1)\big).$$

Conversely, by considering (2.1), we must show that

$$Re\left\{\frac{\Big[z\big((f*g)(z)\big)^{(1+q)}+\lambda z^2\big((f*g)(z)\big)^{(2+q)}\Big]\big(1+\theta e^{i\phi}\big)}{(1-\lambda)\big((f*g)(z)\big)^{(q)}+\lambda z\big((f*g)(z)\big)^{(1+q)}}\right.$$

$$\left.-\frac{\big(\theta e^{i\phi}+\alpha\big)\Big[(1-\lambda)\big((f*g)(z)\big)^{(q)}+\lambda z\big((f*g)(z)\big)^{(1+q)}\Big]}{(1-\lambda)\big((f*g)(z)\big)^{(q)}+\lambda z\big((f*g)(z)\big)^{(1+q)}}\right\}\geq 0. \qquad (2.4)$$

Upon choosing the values of $z$ on the positive real axis where $0\leq z=r<1, Re\big(-e^{i\phi}\big)\geq -\big|e^{i\phi}\big|=-1$ and letting $r\to 1^-$, we conclude to (2.4) by using (2.1) in the left hand of (2.2).

**Corollary 1.** Let $f$ be in the class $WH(n,p,\alpha,\lambda,\theta,q)$. Then

$$a_k \leq \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}, \qquad (2.5)$$

where $p,n \in \mathbb{N}, q \in \mathbb{N}_0, 0 \leq \alpha < p-q, p > q, \theta \geq 0, k \geq n+p,$ and $0 \leq \lambda \leq 1$.

## 3. DISTORTION THEOREM

**Theorem 2.** Let the function $f \in WH(n,p,\alpha,\lambda,\theta,q)$. Then

$$\left[1-\frac{\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^n\right]\frac{p!}{(p-q)!}|z|^{p-q}$$
$$\leq \left|f^{(q)}(z)\right|$$
$$\leq \left[1+\frac{\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^n\right]\frac{p!}{(p-q)!}|z|^{p-q}.$$

The result is sharp for the function $f$ given by (2.2).

**Proof.** Let $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$. Then

$$f^{(q)}(z) = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \delta(k,q)a_k z^{k-q},$$

where

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} 1 & (j=0) \\ i(i-1)\ldots(i-j+1) & (j \neq 0) \end{cases}.$$

Hence

$$f^{(q)}(z) = \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!}a_k z^{k-q}.$$

By (2.5), we get

$$\left|f^{(q)}(z)\right|$$
$$\leq \frac{p!}{(p-q)!}|z|^{p-q}$$
$$+\frac{p!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{(p-q)!\,\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^{n+p-q}$$
$$= \left[1+\frac{\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^n\right]\frac{p!}{(p-q)!}|z|^{p-q}$$

and

$$\left|f^{(q)}(z)\right|$$
$$\geq \left[1-\frac{\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^n\right]\frac{p!}{(p-q)!}|z|^{p-q},$$

when $q=0$, Theorem 2 would provide the growth property of functions in the class $WH(n,p,\alpha,\lambda,\theta,q)$. For $q \in \mathbb{N}$, the results may be looked upon as the distortion properties for the class $WH(n,p,\alpha,\lambda,\theta,q)$.

## 4. CLOSURE THEOREM

Let the functions $f_\iota(z)(\iota = 1,2,\dots,v)$ be defined by

$$f_\iota(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,\iota} z^k \left(a_{k,\iota} \geq 0\right). \tag{4.1}$$

We shall prove the following result for the closure functions in the class $WH(n,p,\alpha,\lambda,\theta,q)$.

**Theorem 3.**Let the functions $f_\iota(z)(\iota = 1,2,\dots,v)$ defined by (4.1) be in the class $WH(n,p,\alpha,\lambda,\theta,q)$. Then the function $h(z)$defined by

$$h(z) = \sum_{\iota=1}^{v} c_\iota f_\iota(z), \quad (c_\iota \geq 0), \tag{4.2}$$

is also in the $WH(n,p,\alpha,\lambda,\theta,q)$, where $\sum_{\iota=1}^{\infty} c_\iota = 1$.

**Proof.** According to the definition of $h(z)$, it can be written as

$$h(z) = \sum_{\iota=1}^{v} c_\iota \left( z^p - \sum_{k=n+p}^{\infty} a_{k,\iota} z^k \right) = \sum_{\iota=1}^{v} c_\iota z^p - \sum_{\iota=1}^{v} \sum_{k=n+p}^{\infty} c_\iota a_{k,\iota} z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \sum_{\iota=1}^{v} c_\iota a_{k,\iota} z^k. \tag{4.3}$$

Furthermore, since the functions $f_\iota(z)(\iota = 1,2,\dots,v)$ are in the class $WH(n,p,\alpha,\lambda,\theta,q)$, then

$$\sum_{k=n+p}^{\infty} \frac{k!\left(1+\lambda(k-q-1)\right)[(k-q)(1+\theta)-(\theta+\alpha)]}{(k-q)!} a_{k,\iota} b_k$$

$$\leq \frac{p!\left(1+\lambda(p-q-1)\right)(p-q-\alpha)}{(p-q)!}.$$

Hence

$$\sum_{k=n+p}^{\infty} \frac{k!\left(1+\lambda(k-q-1)\right)[(k-q)(1+\theta)-(\theta+\alpha)]}{(k-q)!} b_k \left( \sum_{\iota=1}^{v} c_\iota a_{k,\iota} \right)$$

$$= \sum_{\iota=1}^{v} c_\iota \left\{ \sum_{k=n+p}^{\infty} \frac{k!\left(1+\lambda(k-q-1)\right)[(k-q)(1+\theta)-(\theta+\alpha)]}{(k-q)!} b_k a_{k,\iota} \right\}$$

$$\leq \frac{p!\left(1+\lambda(p-q-1)\right)(p-q-\alpha)}{(p-q)!},$$

which implies that $h(z)$be in the class $WH(n,p,\alpha,\lambda,\theta,q)$.

**Corollary 2.**Let the functions $f_\iota(z)$ $(\iota = 1,2)$ defined by (4.1) be in the class $WH(n,p,\alpha,\lambda,\theta,q)$. Then the function $h(z)$defined by

$$h(z) = (1-t)\mathrm{f}_1(z) + t\mathrm{f}_2(z), \quad (0 \le t \le 1), \tag{4.4}$$

is also in the $WH(n,p,\alpha,\lambda,\theta,q)$.

## 5. EXTREME POINTS

We obtain here an extreme points of the class $WH(n,p,\alpha,\lambda,\theta,q)$.

**Theorem 4.** Let $\mathrm{f}_p(z) = z^p$ and

$$\mathrm{f}_k(z) = z^p - \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k} z^k, \tag{5.1}$$

where $k \ge n+p, n,p \in \mathbb{N}, 0 \le \alpha < p-q, p > q, q \in \mathbb{N}_0, \theta \ge 0$ and $0 \le \lambda \le 1$.

Then the function f is in the class $WH(n,p,\alpha,\lambda,\theta,q)$ if and only if it can be expressed in the form:

$$\mathrm{f}(z) = \gamma_p z^p + \sum_{k=n+p}^{\infty} \gamma_k \mathrm{f}_k(z), \tag{5.2}$$

where $\big(\gamma_p \ge 0, \gamma_k \ge 0, k \ge n+p\big)$ and $\gamma_p + \sum_{k=n+p}^{\infty} \gamma_k = 1$.

**Proof.** Suppose that $f$ is expressed in the form (5.2). Then

$$\mathrm{f}(z) = \gamma_p z^p + \sum_{k=n+p}^{\infty} \gamma_k \left[ z^p - \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k} z^k \right]$$

$$= z^p - \sum_{k=n+p}^{\infty} \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k} \gamma_k z^k .$$

Hence

$$\sum_{k=n+p}^{\infty} \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}$$

$$\times \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)\gamma_k}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}$$

$$= \sum_{k=n+p}^{\infty} \gamma_k = 1 - \gamma_p \le 1.$$

Then $f \in WH(n,p,\alpha,\lambda,\theta,q)$.

Conversely, suppose that $f \in WH(n,p,\alpha,\lambda,\theta,q)$. we may set

$$\gamma_k = \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)} a_k,$$

where $a_k$ is given by (2.4). Then

$$\mathrm{f}(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}\gamma_k z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} [z^p - \mathfrak{f}_k(z)]\,\gamma_k = \left(1 - \sum_{k=n+p}^{\infty} \gamma_k\right) z^p + \sum_{k=n+p}^{\infty} \gamma_k\, \mathfrak{f}_k(z) = \gamma_p z^p + \sum_{k=n+p}^{\infty} \gamma_k\, \mathfrak{f}_k(z).$$

This completes the proof of Theorem 4.

## 6. MODIFIED HADAMARD PRODUCT

Let the functions $\mathfrak{f}_\iota(z)$ $(\iota = 1,2)$ defined by (4.1). The modified Hadamard product of the functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ is defined by

$$(\mathfrak{f}_1 * \mathfrak{f}_2)(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1}a_{k,2}z^k. \qquad (6.1)$$

**Theorem 5.** Let the functions $\mathfrak{f}_\iota(z)$ $(\iota = 1,2)$ defined by (4.1) be in the class $WH(n,p,\alpha,\lambda,\theta,q)$and $b_{n+p} \geq b_k$ $(k \geq n+p)$. Then we have $(\mathfrak{f}_1 * \mathfrak{f}_2)(z) \in WH(n,p,\beta,\lambda,\theta,q)$, where $\beta = \frac{(p-q)(n+p)!(p-q)!\left(1+\lambda(n+p-q-1)\right)+p!(n+p-q)!\left(1+\lambda(p-q-1)\right)[\theta-(n+p-q)(1+\theta)]}{(n+p)!(p-q)!\left(1+\lambda(n+p-q-1)\right)-p!(n+p-q)!\left(1+\lambda(p-q-1)\right)}$.

(6.2)

The result is sharp for the functions $\mathfrak{f}_\iota(z)$ given by

$$\mathfrak{f}_\iota(z) = z^p - \frac{p!(n+p-q)!\left(1+\lambda(p-q-1)\right)(p-q-\alpha)}{(n+p)!(p-q)!\left(1+\lambda(n+p-q-1)\right)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}} z^{n+p}, (\iota = 1,2).$$

(6.3)

**Proof.** Employing the technique used earlier by Schild and Silverman (1975), we need to find the largest $\beta = \beta(n,p,\alpha,\lambda,\theta,q)$ such that

$$\sum_{k=n+p}^{\infty} \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)} a_{k,1}a_{k,2} \leq 1. \qquad (6.4)$$

Since the functions $\mathfrak{f}_\iota(z)$ $(\iota = 1,2)$ belong to the class $WH(n,p,\alpha,\lambda,\theta,q)$, then form Theorem 1,we have

$$\sum_{k=n+p}^{\infty} \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)} a_{k,\iota} \leq 1. \qquad (6.5)$$

By the Cauchy Schwarz inequality, we have

$$\sum_{k=n+p}^{\infty} \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \qquad (6.6)$$

Thus, it is sufficient to show that

$$\frac{[(k-q)(1+\theta)-(\theta+\beta)]}{(p-q-\beta)}\sqrt{a_{k,1}a_{k,2}} \le \frac{[(k-q)(1+\theta)-(\theta+\alpha)]}{(p-q-\alpha)},$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{(p-q-\beta)[(k-q)(1+\theta)-(\theta+\alpha)]}{(p-q-\alpha)[(k-q)(1+\theta)-(\theta+\beta)]}. \tag{6.7}$$

But from (6.6), we have

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}, \tag{6.8}$$

consequently, we need only to prove that

$$\frac{(p-q-\alpha)[(k-q)(1+\theta)-(\theta+\beta)]}{(p-q-\beta)[(k-q)(1+\theta)-(\theta+\alpha)]} \le \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)},$$

or equivalently, that

$$\beta \le \frac{(p-q)k!(p-q)!\big(1+\lambda(k-q-1)\big)+p!(k-q)!\big(1+\lambda(p-q-1)\big)[\theta-(k-q)(1+\theta)]}{k!(p-q)!\big(1+\lambda(k-q-1)\big)-p!(k-q)!\big(1+\lambda(p-q-1)\big)}. \tag{6.9}$$

Since the right hand side of (6.9) is an increasing function of $k$ $(k \ge n+p)$.

Hence, we have

$$\beta = \frac{(p-q)(n+p)!\,(p-q)!\,\big(1+\lambda(n+p-q-1)\big)+p!\,(n+p-q)!\,\big(1+\lambda(p-q-1)\big)[\theta-(n+p-q)(1+\theta)]}{(n+p)!\,(p-q)!\,\big(1+\lambda(n+p-q-1)\big)-p!\,(n+p-q)!\,\big(1+\lambda(p-q-1)\big)}.$$

This completes the proof of Theorem 5.

**Theorem 6.** Let the functions $f_\iota(z)$ $(\iota = 1,2)$ defined by (4.1) be in the class $WH(n,p,\alpha,\lambda,\theta,q)$. Then the function

$$h(z) = z^p - \sum_{k=n+p}^{\infty}\big(a_{k,1}^2 + a_{k,2}^2\big)z^k, \tag{6.10}$$

is in the class $WH(n,p,\beta,\lambda,\theta,q)$, where

$$\beta = \frac{(p-q)M + 2N\big(\theta-(n+p-q)(1+\theta)\big)}{M-2N}, \tag{6.11}$$

such that

$$M = (n+p)!\,(p-q)!\,\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]^2 b_{n+p},$$

and

$$N = p!\,(n+p-q)!\,\big(1+\lambda(p-q-1)\big)^2.$$

The result is sharp for the functions $f_\iota(z)(\iota = 1,2)$ given by (6.3).

**Proof.** From Theorem 1, we have

$$\sum_{k=n+p}^{\infty}\left\{\frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}\right\}^2 a_{k,\iota}^2$$

$$\leq\left\{\sum_{k=n+p}^{\infty}\frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}a_{k,\iota}\right\}^2$$

$$\leq 1, (\iota=1,2). \qquad (6.12)$$

It follows that

$$\sum_{k=n+p}^{\infty}\frac{1}{2}\left\{\frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}\right\}^2\left(a_{k,1}^2+a_{k,2}^2\right)$$

$$\leq 1. \qquad (6.13)$$

Therefore, we need to find the largest $\beta$such that

$$\frac{[(k-q)(1+\theta)-(\theta+\beta)]}{(p-q-\beta)}\leq\frac{1}{2}\frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]^2 b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)^2},$$

that is, that

$$\beta\leq\frac{(p-q)M+2N\big(\theta-(k-q)(1+\theta)\big)}{M-2N}, \qquad (6.14)$$

where

$$M=k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]^2 b_k,$$

and

$$N=p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)^2.$$

Since the right hand side of (6.14) is an increasing function of $k$ and $b_{k+1}\geq b_k (k\geq n+p)$, then, setting $k=n+p$ in (6.14), we have (6.11).
This completes the proof of Theorem 6.

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