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# Integral Operator of a New Class of Meromorphic Multivalent Functions

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**Abstract** In this paper, we introduce and study a new class of meromorphic multivalent functions defined by integral operator for this class. We obtain coefficient inequality, like, linear combination, distortion Theorem and Hadamard product (or convolution). Further we obtain a $(n,\delta)$-neighborhood of the function $f \in \vartheta$, and Radius of Starlikeness, Convexity and Close to Convexity.

**Keywords** Meromorphic multivalent functions, Hadamard product (or convolution), Neighborhood, Starlikeness, Convexity, Close to Convexity

## 1. Introduction

Let ב denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \,, (a_k \geq 0, p \in \mathbb{N} = \{1,2,\dots\}). \quad (1.1)$$

Which are analytic and meromorphic multivalent in the punctured unit disk $U^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = U \backslash 0$. Consider a subclass of ב of function of the form:

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \,, (a_k \geq 0, p \in \mathbb{N} = \{1,2,\dots\}). \quad (1.2)$$

The Hadamard product (orconvolution) of two functions,

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} a_k z^k \,, \quad g(z) = z^{-p} + \sum_{k=p}^{\infty} b_k z^k \,,$$

in ב is defined by

$$(f * g)(z) = \mathrm{f(z)} * \mathrm{g(z)} = z^{-p} + \sum_{k=p}^{\infty} a_k b_k z^k \,. \quad (1.3)$$

we shall need to state the extended integral operator for the function belong to the class S which is defined by the following definition.

**Definition (1.1)** The integral operator of $f \in S, for\ \sigma > 0$ is denoted by $I_P^\sigma$ and defined as following:

$$I_P^\sigma f(z) = z^{-p} + \sum_{k=p}^{\infty} \left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^k$$

$$(z \in U^*, P \in \mathbb{N} = \{1,2,\dots\}) \quad (1.4)$$

In particular, when p=1, we have

$$I_1^\sigma f(z) = z^{-1} + \sum_{k=p}^{\infty} \left(\frac{1}{k+2}\right)^{\sigma} a_k z^k \quad (1.5)$$

The integral operator $I_P^\sigma$ was considered, by Uralegaddi and Somantha [10] more recently, Atshan [3] and Srivastava [9].

**Definition (1.2)** Let $f \in$ ב be given by (1.2). Then the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$ is defined by

$$\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda) = \left| \frac{\lambda z^2 \left(I_P^\sigma f(z)\right)'' + \lambda(p+1) z \left(I_P^\sigma f(z)\right)'}{z^2 \left(I_P^\sigma f(z)\right)'' + (\lambda-\alpha) z \left(I_P^\sigma f(z)\right)'} \right| < \beta. \quad (1.6)$$

For $(0 \leq \beta < 1, 0 < \alpha \leq 1, 0 < \lambda < 1, p,k \in \mathbb{N} = \{1,2,\dots\}, z \in U^*)$.

Some other subclass of the $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$ were studied by Cho et al. [5], Atshan and Buti [4], Altintas et al. [1], Liu [8], Joshi et al. [7], Aouf and shammaky [2].

## 2. Coefficients Inequalities

First, in the following theorem, we obtain necessary and sufficient condition for a function $f$ to be in the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$.

**Theorem (2.1)** Let $f \in$ ב be given by (1.2). Then $f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$ if and only if

$$\sum_{k=p+1}^{\infty} \left(\frac{1}{k+p+1}\right)^{\sigma} k(\lambda(k+p) - \beta((k-1) + (\lambda - \alpha a k \leq \beta p p + 1 - \lambda - \alpha, \quad (2.1)$$

for $(0 \leq \beta < 1, 0 < \alpha \leq 1, 0 < \lambda < 1. p,k \in \mathbb{N} = \{1,2,\dots\}, z \in U^*)$

The result is sharp for the function

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$$f(z) = z^{-p} + \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{k+p+1}\right)^{\sigma} k(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)} z^k, k \geq p.$$

**Proof: -**Assume that the inequality (2.1) holds true and let $|z| = 1$, then from (1.6), we have

$$\left|\lambda z^2\big(I_p^\sigma f(z)\big)'' + \lambda(p+1)z\big(I_p^\sigma f(z)\big)'\right| - \beta\left|z^2\big(I_p^\sigma f(z)\big)'' + (\lambda-\alpha)z\big(I_p^\sigma f(z)\big)'\right| \leq 0.$$

$$= \left|\lambda P(P+1)z^{-p} + \lambda\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k(k-1)z^k - \lambda P(p+1)z^{-p} + \lambda(P+1)\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k z^k\right|$$

$$-\beta\left|P(P+1)z^{-p} + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k(k-1)z^k - \lambda P z^{-p} + \lambda\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k z^k + \alpha p z^{-p} - \alpha\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k z^k\right|$$

$$= \left|\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k\, z^k\,\big(\lambda(k-1)+\lambda(p+1)\big)\right|$$

$$-\beta\left|\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k\, z^k\,\big((k-1)+(\lambda-\alpha)\big) + pz^{-p}\big((p+1)-(\lambda-\alpha)\big)\right| \quad (2.2)$$

$$\leq \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))] \leq \beta p\big((p+1)-(\lambda-\alpha)\big).$$

Hence, by maximum modulusprinciple, $f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$.

Conversely, suppose that $f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, then from (2.2), we have

$$\left|\frac{\lambda z^2\big(I_p^\sigma f(z)\big)'' + \lambda(p+1)z\big(I_p^\sigma f(z)\big)'}{z^2\big(I_p^\sigma f(z)\big)'' + (\lambda-\alpha)z\big(I_p^\sigma f(z)\big)'}\right| = \left|\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k\, z^k(\lambda(k+p))}{pz^{-p}\big((p+1)-(\lambda-\alpha)\big) + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k\, z^k\big((k-1)+(\lambda-\alpha)\big)}\right| < \beta.$$

Since $|Re(z)| \leq |z|$ for all $z$, we have

$$Re\left\{\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k\, z^k(\lambda(k+p))}{pz^{-p}\big((p+1)-(\lambda-\alpha)\big) + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k\, z^k\big((k-1)+(\lambda-\alpha)\big)}\right\} < \beta\,, \quad (2.3)$$

We can choose the value of $z$ on the real axis, so that $z^2\big(I_p^\sigma f(z)\big)''$ *and* $z\big(I_p^\sigma f(z)\big)''$ are real, let $z \to 1^-$, through real values . We get the inequality (2.1). Sharpness of the result follows be setting

$$f(z) = z^{-p} + \frac{\beta p\,\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{k+p+1}\right)^{\sigma} k\big[\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\big]} z^k, k \geq p. \quad (2.4)$$

The proof is complete.

**Corollary (2.2)** Let $f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. Then

$$a_k \leq \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{k+p+1}\right)^{\sigma} k\big[\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\big]},$$

Where $(0 \leq \beta < 1, 0 < \alpha \leq 1, 0 < \lambda < 1. p, k \in \mathbb{N} = \{1,2,\ldots\}, z \in U^*)$

# 3. Linear Combination

**Theorem (3.1)** The class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$ is closed convex linear combination.

**Proof**:- Let each of the functions

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_{k,j} z^k \left(a_{k,j} \geq 0, j = 1,2\,, p \in \mathbb{N} = \{1,2,\ldots\}\right), \quad (3.1)$$

be in the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$.it is sufficient to show that the function $h(z)$ defined by

$$h(z)=(1-\epsilon)f_1(z)+\epsilon f_2(z)\in\ \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda).\quad(0\le\epsilon\le1). \tag{3.2}$$

Is also in the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, since

$$h(z)=z^{-p}+\sum_{k=p+1}^{\infty}\ [(1-\epsilon)a_{k,1}+\epsilon a_{k,2}]z^k,\ (0\le\epsilon\le1) \tag{3.3}$$

$$\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma}k\ [\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]\ [(1-\epsilon)a_{k,1}+\epsilon a_{k,2}]$$

$$(1-\epsilon)\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma}k\ [\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]a_{k,1}+\epsilon\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma}k\ [\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]a_{k,2}$$

$$\le(1-\epsilon)\beta p\big((p+1)-(\lambda-\alpha)\big)+\epsilon\beta p\big((p+1)-(\lambda-\alpha)\big)$$

$$=\beta p\big((p+1)-(\lambda-\alpha)\big)\left(0\le\alpha<p,\lambda\ge\frac{1}{p+1},p\in\mathbb{N}=\{1,2,\dots\},n\in\mathbb{N}_0,0\le\epsilon\le1\right) \tag{3.4}$$

Which show that $h(z)\in\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. The proof is complete.

# 4. Neighborhoods

**Definition (4.1)** Let $(0\le\alpha<1,\lambda>-p,0\le r\le1,p\in\mathbb{N}=\{1,2,\dots\},\delta\ge0)$. We define the $\delta-$neighborhoods of a function $f\in T_p$ and denote $N_\delta(f)$ such that

$$N_\delta(f)=\{g\in H: g(z)=z^{-p}+\sum_{k=p}^{\infty}b_kz^k,$$

and

$$\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma}k\ [\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{\beta p\left((p+1)-(\lambda-\alpha)\right)}|a_k-b_k|\le\delta. \tag{4.1}$$

Goodman [6] and Liu [8] have investigate neighborhoods for analytic univalent function we consider this concepts for the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$

**Theorem (4.1)** Let the function $f(z)$ defined by (1.1) be in the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, for every complex number μwith $|\mu|<\delta,\delta\ge0$. Let $\frac{f(z)+\mu z^{-p}}{1+\mu}\in\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, then $N_\delta(f)\subset\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda),\delta\ge0$,

**Proof**:- Since $f\in\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda), f$ satisfies (1.6) and we can write for $\gamma\in\mathbb{C},|\gamma|=1,$ that

$$\left|\frac{\lambda z^2\left(I_p^\sigma f(z)\right)''+\lambda(p+1)z\left(I_p^\sigma f(z)\right)'}{z^2\left(I_p^\sigma f(z)\right)''+(\lambda-\alpha)z\left(I_p^\sigma f(z)\right)'}\right|\ne\gamma. \tag{4.2}$$

Equivalently, we must have

$$\frac{(f*Q)(z)}{z^{-p}}\ne0,z\in U^*, \tag{4.3}$$

Where $Q(z)=z^{-p}+\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma}e_kz^k$,

Such that

$$e_k=\frac{\gamma\left(\frac{1}{k+p+1}\right)^{\sigma}k(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big))}{\beta p\big((p+1)-(\lambda-\alpha)\big)}$$

Satisfying

$$|e_k|\le\frac{\left(\frac{1}{k+p+1}\right)^{\sigma}k(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big))}{\beta p\big((p+1)-(\lambda-\alpha)\big)}$$

Since

$$\frac{f(z)+\mu z^{-p}}{1+\mu}\in\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda),$$

By (4.3)

$$\frac{1}{z^{-p}}\left(\frac{f(z)+\mu z^{-p}}{1+\mu} * Q(z)\right) \neq 0. \qquad (4.4)$$

Now assume that

$\left|\frac{(f*Q)(z)}{z^{-p}}\right| < \delta.$ Then, by (2.3), we have

$$\left|\frac{1}{1+\mu} * \frac{(f*Q)(z)}{z^{-p}} + \frac{\mu}{1+\mu}\right| \geq \frac{|\mu|}{|1+\mu|} - \frac{1}{|1+\mu|} * \left|\frac{(f*Q)(z)}{z^{-p}}\right| > \frac{|\mu|-\delta}{|1+\mu|} \geq 0.$$

That is acontradiction as $|\mu| < \delta$. Therefore

$$\left|\frac{(f*Q)(z)}{z^{-p}}\right| \geq \delta.$$

Letting

$$g(z) = z^{-p} + \sum_{k=p}^{\infty} b_k z^k \in N_\delta(f).$$

Then $\delta - \left|\frac{(g*Q)(z)}{z^{-p}}\right| \leq \left|\frac{(f-g)*Q(z)}{z^{-p}}\right| \leq \left|\sum_{k=p}^{\infty}(a_k - b_k)\, e_k z^k\right| \leq \sum_{k=p}^{\infty}|(a_k - b_k)||e_k||z|^k$

$$< |z|^k \sum_{k=p}^{\infty}\left[\frac{\left(\frac{1}{k+p+1}\right)^{\sigma} k\left(\lambda(k+p) - \beta\big((k-1)+(\lambda-\alpha)\big)\right)}{\beta p\big((p+1)-(\lambda-\alpha)\big)}\right]|(a_k - b_k)| \leq \delta.$$

Therefore,

$\frac{(g*Q)(z)}{z^{-p}} \neq 0$, and we get $g(z) \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, so $N_\delta(f) \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$.

The proof is complete.

# 5. Growth Theorem

**Theorem (5.1)** If the function $f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, then for $0 \leq |z| < 1$.

$$\frac{1}{|z|^p} - \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma}(p+1)\left[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\right]}|z|^{p+1} \leq |f(z)| \leq \frac{1}{|z|^p} + \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma}(p+1)\left[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\right]}|z|^{p+1} \qquad (5.1)$$

The result is sharp and attained for

$$f(z) = \frac{1}{|z|^p} + \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma}[\lambda(1+p)-\beta(\lambda-\alpha)]}|z|^k$$

**Proof:-** Let$f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. Then

$$|f(z)| = \left|z^{-p} + \sum_{k=p}^{\infty} a_k z^k\right| \leq |z|^{-p} + \sum_{k=p}^{\infty} a_k|z|^k \leq |z|^{-p} + |z|\sum_{k=p}^{\infty} a_k.$$

By theorem (2.1), we have

$$\sum_{k=p}^{\infty} a_k \leq \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma}(p+1)\left[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\right]}.$$

Thus

$$|f(z)| \leq \frac{1}{|z|^p} + \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma}(p+1)\left[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\right]}|z|^{p+1}$$

Similarly, we have

$$|f(z)| \geq z^{-p} - \sum_{k=p}^{\infty} a_k |z|^k \geq \frac{1}{z^p} - |z|^{p+1} \sum_{k=p}^{\infty} a_k$$

$$|f(z)| \geq \frac{1}{|z|^p} - \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma} (p+1)\big[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\big]} |z|^{p+1}.$$

Hence result (5.1) follows. The proof is complete.

# 6. Distortion Theorem

**Theorem (6.1)** If the function $f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, then for $0 \leq |z| < 1$.

$$\frac{p}{|z|^{p+1}} - \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma}\big[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\big]} |z|^p \leq |f'(z)| \leq \frac{p}{|z|^{p+1}} + \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma}\big[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\big]} |z|^p \qquad (6.1)$$

With equality for

$$f(z) = \frac{1}{|z|^p} + \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma} (p+1)[\lambda(1+p)-\beta(\lambda-\alpha)]} |z|^{p+1}$$

**Proof:-** Let $f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. Then

$$|f'(z)| \leq \frac{p}{|z|^{p+1}} + \sum_{k=p}^{\infty} k a_k |z|^{k-1} \leq \frac{p}{|z|^{p+1}} + |z|^p \sum_{k=p}^{\infty} (p+1) a_k \leq \frac{p}{|z|^{p+1}} + \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma} \big[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\big]} |z|^p$$

On the other hand

$$|f'(z)| \geq \frac{p}{|z|^{p+1}} - \sum_{k=p}^{\infty} k a_k |z|^{k-1} \geq \frac{p}{|z|^{p+1}} - |z|^p \sum_{k=p}^{\infty} (p+1) a_k \geq \frac{p}{|z|^{p+1}} - \frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{2(p+1)}\right)^{\sigma} \big[\lambda(2p+1)-\beta\big(p+(\lambda-\alpha)\big)\big]} |z|^p.$$

The proof is complete.

# 7. Radius of Starlikeness

**Theorem (7.1)** Let $f \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. Then $f$ is starlike of order $\delta$, $(0 \leq \delta < p)$ in the disc $|z| < r_1$, where

$$r_1(\beta,\sigma,p,k,\alpha,\mu,\lambda) = inf_k \left\{ \frac{(p-\delta)\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{(k-\delta)\,\beta p\big((p+1)-(\lambda-\alpha)\big)} \right\}^{\frac{1}{k+p}}, \qquad (7.1)$$

**Proof:-**It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} + p\right| \leq p-\delta, \qquad (0 < p < \delta),$$

for $|z| < r_1(\beta,\sigma,p,k,\alpha,\mu,\lambda)$.

We have

$$\left|\frac{zf'(z)}{f(z)} + p\right| = \left|\frac{-pz^{-p} + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k z^k}{z^{-p} + \left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^k} + p\right| = \left|\frac{z^{-p}(-p + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k z^{k+p}}{z^{-p}(1 + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^{k+p})} + p\right|$$

$$= \left|\frac{-p + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k z^{k+p} + p + p\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^{k+p}}{1 + \sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^{k+p}}\right|$$

$$\leq \frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^{k+p}(k+p)}{1+\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^{k+p}} \leq p-\delta.$$

If

$$\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^{k+p}(k-\delta)}{\mathrm{p}-\delta} \leq 1. \tag{7.2}$$

Hence, by Theorem (2.1), (7.2) will be true if

$$\frac{(k-\delta)}{\mathrm{p}-\delta}|z|^{k+p} \leq \frac{\left(\frac{1}{k+p+1}\right)^{\sigma} k\left(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\right)}{\beta p\big((p+1)-(\lambda-\alpha)\big)},$$

and hence,

$$|\mathrm{z}| \leq \left\{\frac{(p-\delta)\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{(k-\delta)\,\beta p\big((p+1)-(\lambda-\alpha)\big)}\right\}^{\frac{1}{k+p}}$$

The proof is complete.

# 8. Radius of Convexity

**Theorem (8.1**) If $f(z) \in r_2(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. Then $f(z)$ is p-valent convex of order $\delta, (0 \leq \delta < p)$ in the disc $|z| < r_2$, where

$$r_2(\beta,\sigma,p,k,\alpha,\mu,\lambda) = inf_k \left\{\frac{p(p-\delta)\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{k(k+\delta)\,\beta p\,((p+1)-(\lambda-\alpha))}\right\}^{\frac{1}{k+p}}, \tag{8.1}$$

**Proof:-**It is sufficient to show that

$$\left|1+\frac{zf''(z)}{f'(z)}+p\right| \leq p-\delta, \qquad (0 \leq \delta < p)$$

for $|z| < r_2(\beta,\sigma,p,k,\alpha,\mu,\lambda)$.

We have

$$\left|1+\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}+\mathrm{p}\right| = \left|\frac{\mathrm{p}(\mathrm{p}+1)\mathrm{z}^{-\mathrm{p}-1}+\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma} \mathrm{a_k k(k-1)z^{k-1}}}{-\mathrm{pz}^{-\mathrm{p}-1}+\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma}\mathrm{a_k k z^{k-1}}}+(1+\mathrm{p})\right|$$

$$= \left|\frac{\mathrm{z}^{-\mathrm{p}-1}\left[\mathrm{p}(\mathrm{p}+1)+\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma}\mathrm{a_k k(k-1)z^{k+p}}\right]}{\mathrm{z}^{-\mathrm{p}-1}\left[-\mathrm{p}+\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma}\mathrm{a_k k z^{k+p}}\right]}+(1+\mathrm{p})\right|$$

$$\left|\frac{\left[\mathrm{p}(\mathrm{p}+1)+\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma}\mathrm{a_k k(k-1)z^{k+p}}\right]-\mathrm{p}(\mathrm{p}+1)+(\mathrm{p}+1)\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma}\mathrm{a_k k z^{k+p}}}{-\mathrm{p}+\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma}\mathrm{a_k k z^{k+p}}}\right|$$

$$\left|\frac{\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma}\mathrm{a_k k(k+p)z^{k+p}}}{-\mathrm{p}+\sum_{\mathrm{k=p}}^{\infty}\left(\frac{1}{\mathrm{k+p+1}}\right)^{\sigma}\mathrm{a_k k z^{k+p}}}\right|$$

$$\leq \frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k(k+p)|z|^{k+p}}{-p+\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k|z|^{k+p}} \leq p-\delta.$$

If

$$\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k k(k+\delta)|z|^{k+p}}{p(p-\delta)} \leq 1. \tag{8.2}$$

Hence, by Theorem (2.3), (6.2) will be true if

$$\frac{k(k+\delta)}{\mathrm{p}(\mathrm{p}-\delta)}|z|^{k+p} \leq \frac{\left(\frac{1}{k+p+1}\right)^{\sigma} k\left(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\right)}{\beta p\big((p+1)-(\lambda-\alpha)\big)},$$

and hence,

$$|\mathrm{z}| \leq \left\{\frac{p(p-\delta)\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{k(k+\delta)\,\beta p\big((p+1)-(\lambda-\alpha)\big)}\right\}^{\frac{1}{k+p}}$$

The proof is complete.

# 9. Hadamard Product

**Theorem (9.1)** Let $f, g \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. Then $f*g \in \zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$ for

$$f(z)=z^{-p}+\sum_{k=n+1}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k z^k \quad , \quad g(z)=z^{-1}+\sum_{k=n+1}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} b_k z^k$$

and

$$(f*g)(z)=z^{-p}+\sum_{k=n+1}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} a_k b_k z^k,$$

where

$$\delta=\frac{\beta^2 p\big((p+1)-(\lambda-\alpha)\big)\big(k\lambda(k+p)\big)}{\beta^2 p\big((p+1)-(\lambda-\alpha)\big)((k-1)+(\lambda-\alpha))+\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]^2}.$$

**Proof:-**Since $f$ and $g$ are in the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, then

$$\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]a_k}{\beta p\big((p+1)-(\lambda-\alpha)\big)} \leq 1, \tag{9.1}$$

and

$$\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]b_k}{\beta p\big((p+1)-(\lambda-\alpha)\big)} \leq 1. \tag{9.2}$$

We have to find the largest $\delta$ such that

$$\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{\beta p\big((p+1)-(\lambda-\alpha)\big)} a_k b_k \leq 1. \tag{9.3}$$

By Cauchy-Schwarz inequality, we get

$$\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{\beta p\big((p+1)-(\lambda-\alpha)\big)} \sqrt{a_k b_k} \leq 1. \tag{9.4}$$

We want only to show that

$$\frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{\beta p\big((p+1)-(\lambda-\alpha)\big)} a_k b_k$$

$$\leq \frac{\sum_{k=p}^{\infty}\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]}{\beta p\big((p+1)-(\lambda-\alpha)\big)} \sqrt{a_k b_k}\,.$$

This equivalently to

$$\sqrt{a_k b_k} \leq \frac{\delta\big[k\big(\lambda(k+p)\big)-\beta\big((k-1)+(\lambda-\alpha)\big)\big]}{\beta\big[k\big(\lambda(k+p)\big)-\delta\big((k-1)+(\lambda-\alpha)\big)\big]}\,. \quad (9.5)$$

From (9.4), we get

$$\sqrt{a_k b_k} \leq \frac{\beta p\,\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{k+p+1}\right)^{\sigma} k\Big(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\Big)}. \quad (9.6)$$

Thus it is enough to show that

$$\frac{\beta p\big((p+1)-(\lambda-\alpha)\big)}{\left(\frac{1}{k+p+1}\right)^{\sigma} k\Big(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\Big)} \leq \frac{\delta\big[k\big(\lambda(k+p)\big)-\beta\big((k-1)+(\lambda-\alpha)\big)\big]}{\beta\big[k\big(\lambda(k+p)\big)-\delta\big((k-1)+(\lambda-\alpha)\big)\big]},$$

which simplifies to

$$\delta = \frac{\beta^2 p\big((p+1)-(\lambda-\alpha)\big)\big(k\lambda(k+p)\big)}{\beta^2 p\big((p+1)-(\lambda-\alpha)\big)((k-1)+(\lambda-\alpha))+\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]^2}.$$

The proof is complete.

**Theorem (10.1)** Let the function $f_j\,(j=1,2)$ defined by $f_j(z)=z^{-p}+\sum_{k=n+1}^{\infty} a_{k,j} z^k\,,(a_{k,j}\geq 0, j=1,2)$ be in the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. Then the function $h$ defined by

$$h(z)=z^{-p}+\sum_{k=p}^{\infty}({a_{k,1}}^2+{a_{k,2}}^2)z^k\ , \quad (10.1)$$

belongs to the $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$ where

$$\eta = \frac{\beta^2 p^2\big((p+1)-(\lambda-\alpha)\big)\big(k\lambda(k+p)\big)}{\beta^2 p^2\big((p+1)-(\lambda-\alpha)\big)((k-1)+(\lambda-\alpha))+\left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]^2}\ .$$

**Proof:-**Note that

$$\sum_{k=p}^{\infty}\left[\frac{\left(\frac{1}{k+p+1}\right)^{\sigma} k\Big(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\Big)}{\beta p\big((p+1)-(\lambda-\alpha)\big)}\right]^2 {a_{k,j}}^2$$

$$\leq \left[\sum_{k=p}^{\infty}\frac{\left(\frac{1}{k+p+1}\right)^{\sigma} k\Big(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\Big)}{\beta p\,\big((p+1)-(\lambda-\alpha)\big)} a_{k,j}\right]^{\mathbf{2}} \leq 1\ .(j=1,2) \quad (10.2)$$

For $f_j\in\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, we have

$$\sum_{k=p}^{\infty}\ \frac{1}{2}\left(\frac{\left(\frac{1}{k+p+1}\right)^{\sigma} k\Big(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\Big)}{\beta p\,\big((p+1)-(\lambda-\alpha)\big)}\right)^2\big({a_{k,1}}^2+{a_{k,2}}^2\big)\leq 1\,. \quad (10.3)$$

In order to obtain our result we have to find the largest $\eta$ such that

$$\frac{\left(\frac{1}{k+p+1}\right)^{\sigma} k\Big(\lambda(k+p)-\eta\big((k-1)+(\lambda-\alpha)\big)\Big)}{\eta} \leq \frac{\left(\frac{1}{k+p+1}\right)^{\sigma} k\Big(\lambda(k+p)-\beta\big((k-1)+(\lambda-\alpha)\big)\Big)}{\beta^2 p^2\big((p+1)-(\lambda-\alpha)\big)}.\ \ (k\geq p)$$

So that

$$\eta = \frac{\beta^2 p^2\big((p+1)-(\lambda-\alpha)\big)\big(k\lambda(k+p)\big)}{\beta^2 p^2\big((p+1)-(\lambda-\alpha)\big)((k-1)+(\lambda-\alpha)) + \left(\frac{1}{k+p+1}\right)^{\sigma} k\,[\lambda(k+p)-\beta((k-1)+(\lambda-\alpha))]^2} .$$

The proof is complete.

# 10. Integral Operator

**Theorem (11.1)** If $f$ be in the class $\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$. Then the integral operator

$$F(z) = c\int_0^1 u^c f(uz)\,du\,,\ (0<u\le 1\,,0<c<\infty), \tag{11.1}$$

is in the class $\zeta\left(\alpha,\mu,\lambda,\frac{1+\beta c}{2+c}\right)$.

The result is sharp for the function

$$f_k(z) = z^{-p} + \frac{\left(\frac{1+\beta c}{2+c}\right)p((p+1)-(\lambda-\alpha))}{k(\lambda(k+p)-(\frac{1+\beta c}{2+c})((k-1)+((\lambda-\alpha)(\lambda-\alpha))} z^{\,k}.$$

**Proof:-**By definition of $F$, we get

$$F(z) = c\int_0^1 u^c f(uz)\,du$$

$$= c\int_0^1 \left(u^{c-1}z^{-p} + \sum_{k=p}^{\infty} a_k u^{k+c} z^k\right) du$$

$$= z^{-p} + \sum_{k=p}^{\infty} \frac{c}{k+c+1} a_k z^k \ . \tag{11.2}$$

By Theorem (2.1), It is sufficient to show that

$$\sum_{k=p+1}^{\infty} \frac{c\left(\frac{1}{k+p+1}\right)^{\sigma} k\left(\lambda(k+p)-\left(\frac{1+\beta c}{2+c}\right)((k-1)+(\lambda-\alpha))\right)}{(c+k+1)\left(\frac{1+\beta c}{2+c}\right)p\big((p+1)-(\lambda-\alpha)\big)}. \tag{11.3}$$

Since, if $f\in\zeta(\beta,\sigma,p,k,\alpha,\mu,\lambda)$, then (10.1) satisfies if

$$\frac{c}{(c+k+1)\left(\frac{1+\beta c}{2+c}\right)p\big((p+1)-(\lambda-\alpha)\big)} \le \frac{1}{\beta p\big((p+1)-(\lambda-\alpha)\big)}.$$

Oe equivalently, when

$$\emptyset(k,c,\beta) = \frac{c\,\beta p\big((p+1)-(\lambda-\alpha)\big)}{(c+k+1)\left(\frac{1+\beta c}{2+c}\right)p\big((p+1)-(\lambda-\alpha)\big)} \le 1.$$

Since $\emptyset(k,c,\beta)$ is a decreasing function of $k,$ and the result is sharp for

$$f_k(z) = z^{-p} + \frac{\left(\frac{1+\beta c}{2+c}\right)p((p+1)-(\lambda-\alpha))}{k(\lambda(k+p)-(\frac{1+\beta c}{2+c})((k-1)+((\lambda-\alpha)(\lambda-\alpha))} z^{\,k}.$$

The proof is complete.

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