American Journal of Computational and Applied Mathematics 2016, 6(1): 1-6
DOI: 10.5923/j.ajcam.20160601.01

On a Certain Subclass of Harmonic Multivalent Functions

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Abstract In this paper, we introduce a certain subclass of harmonic multivalent functions in the unit disk U=$\{z \epsilon C : |z| < 1\}$. We obtain some interesting properties, like, coefficient conditions, convex set, distortion theorems, weighted mean.

Keywords Harmonic multivalent functions, Distortion theorem, Convex set

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset C$ if both u and v are real harmonic in D. In any simply connected domain D, we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. The harmonic function $f = h + \bar{g}$ is sense-preserving and locally one-to-one in D if $|h'(z)| > |g'(z)|$ inD. See Clunie and Sheil-Small [3].

For $p \geq 1, k \in N$, denote by $G(k,p)$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disk U=$\{z \epsilon C : |z| < 1\}$, where h and g are defined by

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \; g(z) = \sum_{k=p}^{\infty} b_k z^k, |b_1| < 1 \quad (1)$$

which are analytic and multivalent functions in U.

Also, denote by $\bar{G}(k,p)$ subclass of $G(k,p)$ consisting of harmonic functions $f = h + \bar{g}$ where h and g are of the form

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k,$$
$$g(z) = -\sum_{k=p}^{\infty} |b_k| z^k, \; |b_1| < 1 \quad (2)$$

In 1984, Clunie and Sheil-Small [3] investigated the class S_H and as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. For more basic results one may refer to the following standard introductory text book by Duren [4], see also Ahuja [1], Jahangiri et al. [5] and Ponnusamy and Rasila ([6], [7]).

The convolution of two functions of form

$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and $F(z) = z^p + \sum_{k=p+1}^{\infty} A_k z^k$ is defined as

$$(f * F)(z) = f(z) * F(z) = z^p + \sum_{k=2}^{\infty} a_k A_k z^k \quad (3)$$

Let $Gp(k,p,\lambda,\beta,\mu)$ denote the subclass of $G(k,p)$ consisting of functions $f = h + \bar{g} \in G(k,p)$ that satisfy the condition

$$\mathrm{Re}\left\{(1-\lambda) + \frac{\beta\lambda z^p + (1-\lambda) z^2 (f(z))''}{z(f(z))'}\right\} \quad (4)$$
$$> (\beta\lambda + (1-\lambda)p^2)\mu, 0 \leq \lambda \leq 1, \mu < \frac{1}{p}, \beta \geq 0, p \in N = \{1,2,\ldots\}$$

Define $\overline{Gp}(k,p,\lambda,\beta,\mu) = Gp(k,p,\lambda,\beta,\mu) \cap \bar{G}(k,p)$.

Lemma (1) [2]: Let $\alpha \geq 0$. Then $Re\{w\} > \alpha$ if and only if $|w - (1+\alpha)| < |w + (1-\alpha)|$, where w be any complex number.

2. Coefficient Estimates

We begin with a sufficient condition for the function in$f$ to be the class $Gp(k,p,\lambda,\beta,\mu)$.

Theorem (2.1): Let $f = h + \bar{g}$ defined in (1). If

$$\sum_{k=p+1}^{\infty}\left[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k\right]|a_k| + \sum_{k=p}^{\infty}\left[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k\right]|b_k| - p \leq 0, \quad (5)$$

Where $0 \leq \lambda \leq 1$, $\mu < \frac{1}{p}$, $p \geq 2$, then f is harmonic Multivalent sense-preserving in U and $f \in Gp(k,p,\lambda,\beta,\mu)$.

Proof: Let

$$w(z) = \left\{(1-\lambda) + \frac{\beta\lambda z^p + (1-\lambda)z^2(f(z))''}{z(f(z))'}\right\} = \frac{A(z)}{B(z)}$$

By Lemma (1), we must show that (4) holds true. It suffices to show that

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Published online at http://journal.sapub.org/ajcam

$$|A(z)-(1+(\beta\lambda+(1-\lambda)p^2)\mu\,)B(z)| - |A(z) + (1-(\beta\lambda+(1-\lambda)p^2)\mu\,)B(z)| \le 0$$

Substituting for w and resorting to simple calculation, we find

$$|A(z)-(1+(\beta\lambda+(1-\lambda)p^2)\mu\,)B(z)| =$$

$$= \left|(1-\lambda)z(f(z))' + \beta\lambda z^p + (1-\lambda)\left[z^2(f(z))''\right] - (1+(\beta\lambda+(1-\lambda)p^2)\mu)[z(f(z))']\right|$$

$$= \left|(1-\lambda)z\left[(h(z))' + \overline{(g(z))'}\right] + \beta\lambda z^p + (1-\lambda)z^2\left[(h(z))'' + \overline{(g(z))''}\right] - (1+(\beta\lambda+(1-\lambda)p^2)\mu)z\left[(h(z))' + \overline{(g(z))'}\right]\right|$$

$$= \left|(1-\lambda)pz^p + (1-\lambda)\sum_{k=p+1}^{\infty} ka_k z^k + (1-\lambda)\sum_{k=p}^{\infty} kb_k z^k + \beta\lambda z^p + (1-\lambda)p(p-1)z^p + (1-\lambda)\sum_{k=p+1}^{\infty} k(k-1)a_k z^k + (1-\lambda)\sum_{k=p}^{\infty} k(k-1)b_k z^k - (1+(\beta\lambda+(1-\lambda)p^2)\mu)pz^p - (1+(\beta\lambda+(1-\lambda)p^2)\mu)\sum_{k=p+1}^{\infty} ka_k z^k - (1+(\beta\lambda+(1-\lambda)p^2)\mu)\sum_{k=p}^{\infty} kb_k z^k\right|$$

$$= |[(1-\lambda)p + (1-\lambda)p(p-1) + \beta\lambda - (1+(\beta\lambda+(1-\lambda)p^2)\mu)p]z^p + \sum_{k=p+1}^{\infty}[(1-\lambda)k + (1-\lambda)k(k-1) - (1+(\beta\lambda+(1-\lambda)p^2)\mu)k]a_k z^k + \sum_{k=p}^{\infty}[(1-\lambda)k + (1-\lambda)k(k-1) - (1+(\beta\lambda+(1-\lambda)p^2)\mu)k]b_k z^k|$$

$$= |[(\beta\lambda+(1-\lambda)p^2)(1-p\mu) - p]z^p + \sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k - k]a_k z^k + \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k - k]b_k z^k\,|$$

$$\le [(\beta\lambda+(1-\lambda)p^2)(1-p\mu) - p]|z|^p + \sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k - k]|a_k||z|^k + \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k - k]|b_k||z|^k$$

$$|A(z) + (1-(\beta\lambda+(1-\lambda)p^2)\mu)B(z)| =$$

$$= \left|(1-\lambda)z(f(z))' + \beta\lambda z^p + (1-\lambda)\left[z^2(f(z))''\right] + (1-(\beta\lambda+(1-\lambda)p^2)\mu)[z(f(z))']\right|$$

$$= \left|(1-\lambda)z\left[(h(z))' + \overline{(g(z))'}\right] + \beta\lambda z^p + (1-\lambda)z^2\left[(h(z))'' + \overline{(g(z))''}\right] + (1-(\beta\lambda+(1-\lambda)p^2)\mu)z\left[(h(z))' + \overline{(g(z))'}\right]\right|$$

$$= \Bigg|(1-\lambda)pz^p + (1-\lambda)\sum_{k=p+1}^{\infty} ka_k z^k + (1-\lambda)\sum_{k=p}^{\infty} kb_k z^k + \beta\lambda z^p + (1-\lambda)p(p-1)z^p + (1-\lambda)\sum_{k=p+1}^{\infty} k(k-1)a_k z^k + (1-\lambda)\sum_{k=p}^{\infty} k(k-1)b_k z^k + (1-(\beta\lambda+(1-\lambda)p^2)\mu)pz^p + (1-(\beta\lambda+(1-\lambda)p^2)\mu)\sum_{k=p+1}^{\infty} ka_k z^k + (1-(\beta\lambda+(1-\lambda)p^2)\mu)\sum_{k=p}^{\infty} kb_k z^k\Bigg|$$

$$= |[(1-\lambda)p + (1-\lambda)p(p-1) + \beta\lambda + (1-(\beta\lambda+(1-\lambda)p^2)\mu)p]z^p + \sum_{k=p+1}^{\infty}[(1-\lambda)k + (1-\lambda)k(k-1) + (1-(\beta\lambda+(1-\lambda)p^2)\mu)k]a_k z^k + \sum_{k=p}^{\infty}[(1-\lambda)k + (1-\lambda)k(k-1) + (1-(\beta\lambda+(1-\lambda)p^2)\mu)k]b_k z^k|$$

$$= |[(\beta\lambda+(1-\lambda)p^2)(1-p\mu)+p]z^p + \sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]a_k z^k + \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]b_k z^k|$$

$$\geq [(\beta\lambda+(1-\lambda)p^2)(1-p\mu)+p]|z|^p - \sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]|a_k||z|^k - \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]|b_k||z|^k$$

$$|A(z) - (1+(\beta\lambda+(1-\lambda)p^2)\mu)B(z)| - |A(z) + (1-(\beta\lambda+(1-\lambda)p^2)\mu\,)B(z)|$$

$$\leq [(\beta\lambda+(1-\lambda)p^2)(1-p\mu)-p] + \sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k - k]|a_k| + \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k - k]|b_k|$$

$$- \Bigg[[(\beta\lambda+(1-\lambda)p^2)(1-p\mu)+p] - \sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]|a_k| - \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k + k]|b_k|\Bigg]$$

$$= -2p + 2\sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|a_k| + 2\sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|b_k|$$

$$\sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|a_k| + \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|b_k| - p \leq 0$$

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{X_k}{[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]} z^k + \sum_{k=p}^{\infty} \frac{\overline{Y_k}}{\lambda[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]} (\bar{z})^k,$$

where

$$\sum_{k=p+1}^{\infty} |X_k| + \sum_{k=p}^{\infty} \overline{|Y_k|} = p$$

$$\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|a_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|b_k|$$

$$= \sum_{k=p+1}^{\infty} |X_k| + \sum_{k=p}^{\infty} \overline{|Y_k|} = p$$

Theorem (2.2): Let $f = h + \bar{g}$ with h and g given by (2). Then $f \in \overline{Gp}(k, p, \lambda, \beta, \mu)$ if and only if

$$\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|a_k| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|b_k| \leq p \,. \quad (6)$$

Proof: From theorem (2.1) to prove the necessary part, let us assume that $f \in \overline{Gp}(k, p, \lambda, \beta, \mu)$ using (4), we get

$$Re\left\{(1-\lambda) + \frac{\beta\lambda z^p + (1-\lambda)z^2(f(z))''}{z(f(z))'}\right\}$$

$$= Re\left\{\frac{(1-\lambda)z(f(z))' + \beta\lambda z^p + (1-\lambda)\, z^2(f(z))''}{z(f(z))'}\right\}$$

$$= Re\left\{\frac{(1-\lambda)z\left[(h(z))' + \overline{(g(z))'}\right] + \beta\lambda z^p + (1-\lambda)z^2\left[(h(z))'' + \overline{(g(z))''}\right]}{z\left[(h(z))' + \overline{(g(z))'}\right]}\right\}$$

$$= Re\left\{\frac{[(1-\lambda)p^2 + \beta\lambda]z^p - \sum_{k=p+1}^{\infty}[(1-\lambda)k^2]|a_k|z^k - \sum_{k=p}^{\infty}[(1-\lambda)k^2]|b_k|(\bar{z})^k}{pz^p - \sum_{k=p+1}^{\infty} k|a_k|z^k - \sum_{k=p}^{\infty} k|b_k|(\bar{z})^k}\right\}$$

$$> (\beta\lambda + (1-\lambda)p^2)\mu \,.$$

If we choose z to be real and let $z \to 1^-$, we obtain the condition (6) and the proof is complete.

3. Convex Set

Theorem (3.1): The class $\overline{Gp}(k, p, \lambda, \beta, \mu)$ is convex set.

Proof: Let the function $f_i(z)(i = 1,2)$ be in the class $\overline{Gp}(k, p, \lambda, \beta, \mu)$. It is sufficient to show that the function H defined by:

$$H(z) = (1-e)f_1(z) + ef_2(z)\,, \ (0 \leq e \leq 1) \quad (7)$$

is in the class $\overline{Gp}(k, p, \lambda, \beta, \mu)$ we have

$$f_i = z^p + \sum_{k=p+1}^{\infty} |a_{k,i}| z^k + \overline{\sum_{k=p}^{\infty} |b_{k,i}| z^k}\,.$$

Since for e $(0 \leq e \leq 1)$

$$H(z) = z^p + \sum_{k=p+1}^{\infty} ((1-e)|a_{k,1}| - e|a_{k,2}|)z^k + \sum_{k=p}^{\infty} ((1-e)|b_{k,1}| - e|b_{k,2}|)\bar{z}^k\,.$$

In view of Theorem (2.1), we have

$$\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]\left((1-e)|a_{k,1}| - e|a_{k,2}|\right) + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]\left((1-e)|b_{k,1}| - e|b_{k,2}|\right)$$

$$= (1-e)\left[\sum_{k=p+1}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|a_{k,1}| + \sum_{k=p}^{\infty} [(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|b_{k,1}|\right]$$

$$+e\left[\sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu)-\beta\lambda\mu k]\left|a_{k,2}\right|+\sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu)-\beta\lambda\mu k]\left|b_{k,2}\right|\right]$$
$$\leq (1-e)p+ep=p\,.$$

Hence, $H(z)\in\overline{Gp}(k,p,\lambda,\beta,\mu)$. This completes the proof.

4. Distortion and Covering

The next Theorem is on the distortion and covering bounds for functions in the class $\overline{Gp}(k,p,\lambda,\beta,\mu)$.

Theorem (4.1): If $f\in\overline{Gp}(k,p,\lambda,\beta,\mu), |z|=r<1$, then

$$|f(z)|\geq\left(1-|b_p|\right)r^p-\frac{p[1-|b_p|]}{(p+1)[(1-\lambda)(1+p-p^2\mu)-\beta\lambda\mu]}r^{p+1}, \tag{8}$$

and

$$|f(z)|\leq\left(1+|b_p|\right)r^p+\frac{p[1-|b_p|]}{(p+1)[(1-\lambda)(1+p-p^2\mu)-\beta\lambda\mu]}r^{p+1}\,. \tag{9}$$

Proof: Assume that $f\in\overline{Gp}(k,p,\lambda,\beta,\mu)$. Then by Theorem (2.2), we obtain

$$|f(z)|=\left|z^p+\sum_{k=p+1}^{\infty}a_kz^k+\sum_{k=p}^{\infty}b_k(\bar z)^k\right|\geq\left(1-|b_p|\right)r^p-\sum_{k=p+1}^{\infty}(|a_k|+|b_k|)r^k$$
$$\geq\left(1-|b_p|\right)r^p-\sum_{k=p+1}^{\infty}(|a_k|+|b_k|)r^{p+1}$$
$$=\left(1-|b_p|\right)r^p-\frac{1}{(p+1)[(1-\lambda)(1+p-p^2\mu)-\beta\lambda\mu]}\sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu)-\beta\lambda\mu k](|a_k|+|b_k|)r^{p+1}$$
$$\geq\left(1-|b_p|\right)r^p-\frac{1}{(p+1)[(1-\lambda)(1+p-p^2\mu)-\beta\lambda\mu]}\sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu)-\beta\lambda\mu k](|a_k|+|b_k|)r^{p+1}$$
$$\geq\left(1-|b_p|\right)r^p-\frac{1}{(p+1)[(1-\lambda)(1+p-p^2\mu)-\beta\lambda\mu]}\left[p-p|b_p|\right]r^{p+1}$$
$$=\left(1-|b_p|\right)r^p-\frac{p(p-1)\left[1-|b_p|\right]}{(p+1)[(1-\lambda)(1+p-p^2\mu)-\beta\lambda\mu]}r^{p+1}\,.$$

Relation (9) can be proved by using the similar statements.

The covering result given in corollary (4.2) follows from the inequality (9) of this Theorem.

Corollary (4.2): If $f\in\overline{Gp}(k,p,\lambda,\beta,\mu)$, then

$$\left\{w:|w|<\left(1-|b_p|\right)r^p-\frac{p\left[1-|b_p|\right]}{\left[(p+1)[(1-\lambda)(1+p-p^2\mu)-\beta\lambda\mu]\right]}\right\}\subset f(U)\,.$$

5. Weighted Mean

Definition (5.1): Let $f\ and\ F\ \epsilon\ \overline{Gp}(k,p,\lambda,\beta,\mu)$, where

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k+\sum_{k=p}^{\infty}b_kz^k\ \ and\ F(z)=z^p+\sum_{k=p+1}^{\infty}A_kz^k+\sum_{k=p}^{\infty}B_kz^k\,.$$

Then the weighted mean $E_i(z)$ of f and g is given by

$$E_i(z)=\frac{1}{2}[(1-i)f(z)+(1+i)F(z)],\ \ 0<i<1\,.$$

In the theorem below, we will show the weighted mean for this class:

Theorem (5.2): If f and F be in the class $\overline{Gp}(k,p,\lambda,\beta,\mu)$, then the weighted mean of f and F is also in the class $\overline{Gp}(k,p,\lambda,\beta,\mu)$.

Proof: By Definition (5.1), we have

$$E_i(z) = \frac{1}{2}[(1-i)f(z) + (1+i)g(z)]$$

$$E_i(z) = \frac{1}{2}\left[(1-i)\left(z^p + \sum_{k=p+1}^{\infty} a_k z^k + \sum_{k=p}^{\infty} b_k z^k\right) + (1+i)\left(z^p + \sum_{k=p+1}^{\infty} A_k z^k + \sum_{k=p}^{\infty} B_k \bar{z}^k\right)\right]$$

$$= \frac{1}{2}\left[\left(z^p + \sum_{k=p+1}^{\infty} (1-i)a_k z^k + \sum_{k=p}^{\infty} (1-i) b_k z^k\right) + \left(z^p + \sum_{k=p+1}^{\infty} (1+i)A_k z^k + \sum_{k=p}^{\infty} (1+i) B_k \bar{z}^k\right)\right]$$

$$= z^p + \sum_{k=p+1}^{\infty} \frac{1}{2}[(1-i)a_k + (1+i)A_k] z^k + \sum_{k=p}^{\infty} \frac{1}{2}[(1-i)b_k + (1+i)B_k]\bar{z}^k .$$

Since f and F are in the class $\overline{Gp}(k,p,\lambda,\beta,\mu)$ so by Theorem(2.2), we get

$$\frac{\sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|a_k| + \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|b_k|}{p} \le 1$$

and

$$\frac{\sum_{k=p+1}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|A_k| + \sum_{k=p}^{\infty}[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]|B_k|}{p} \le 1$$

Then

$$= \sum_{k=p+1}^{\infty} \frac{[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]}{p}\left(\frac{1}{2}[(1-i)|a_k| + (1+i)|A_k|]\right)$$

$$+ \sum_{k=p}^{\infty} \frac{[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]}{p}\left(\frac{1}{2}[(1-i)|b_k| + (1+i)|B_k|]\right)$$

$$= \frac{1}{2}\sum_{k=p+1}^{\infty} \frac{[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]}{p}\big((1-i)|a_k| + (1+i)|A_k|\big)$$

$$+ \frac{1}{2}\sum_{k=p}^{\infty} \frac{[(1-\lambda)k(k-p^2\mu) - \beta\lambda\mu k]}{p}\big((1-i)|b_k| + (1+i)|B_k|\big)$$

$$\le \frac{1}{2}(1-i) + \frac{1}{2}(1+i) + \frac{1}{2}(1-i) + \frac{1}{2}(1+i) = 2 .$$

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