

Anti Fuzzy Bi- Γ - Ideals Of Γ - Semigroups

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Abstract

In this paper , we defined the algebraic structures of anti fuzzy bi- Γ -ideal , anti Q- fuzzy bi- Γ -ideal and studied some basic properties .

Introduction

In 1965 Zadeh introduced the concepts of fuzzy set [6]. In 1980 Kuroki introduced and studied fuzzy ideals and fuzzy bi-ideals in semigroup [8]. In 1986 Sen and Saha gave a characterization of a bi- Γ -semigroup [7]. In 2007 Chinram and Jirojkal introduced the concept of a bi- Γ -ideal in a Γ -semigroup [9]. In 2009 Williams, Latha and Chandrasekeran introduced a notion of fuzzy bi- Γ -ideal of a Γ -semigroup [3].

In this paper we define anti fuzzy subsemigroup , anti fuzzy bi-ideal , anti fuzzy bi- Γ -ideal , anti fuzzy bi- Γ -ideal of Γ - semigroup and study some of their related properties .

1-Preliminaries

In this section we review some basic definitions which will be used in this paper .

Definition (1.1) [6]: A function μ from a non empty set X to the interval $[0,1]$ is called a fuzzy set of X and the complement of μ is denoted μ^c is a fuzzy set of X defined by $\mu^c(x)=1-\mu(x) \forall x \in X$.

Definition (1.2) [4]: Let f be a function from a set X to a set Y while μ is a fuzzy set of X then the image $f(\mu)$ of μ is the fuzzy set $f(\mu): Y \rightarrow [0,1]$, defined by

$$f(\mu(y)) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

Definition (1.3) [4]: Let f be a function from a set X to a set Y while μ is fuzzy set of Y then the inverse image $f^{-1}(\mu)$ of μ under f is the fuzzy set $f^{-1}(\mu): X \rightarrow [0,1]$, defined by $f^{-1}(\mu)(x) = \mu(f(x))$.

Definition (1.4) [4]: Let σ and μ be two fuzzy set of a sets X then the union of μ and σ denoted by $\mu \cup \sigma$ is a fuzzy set of X defined by $(\mu \cup \sigma)(x) = \max \{ \mu(x), \sigma(x) \}$ and the intersection of μ and σ denoted by $\mu \cap \sigma$ is a fuzzy set of X defined by $(\mu \cap \sigma)(x) = \min \{ \mu(x), \sigma(x) \}$.

Definition (1.5) [4]: Let μ be a fuzzy set of a set X and $t \in [0,1]$. The subset μ_t of X define by $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a level set of X .

Definition (1.6) [4]: Let μ be a fuzzy set of a set X and $t \in [0,1]$. The subset $L(\mu, t)$ of X define by $L(\mu, t) = \{x \in X \mid \mu(x) \leq t\}$ is called a lower level set of X .

Definition (1.7) [2]: A fuzzy set μ of a set X is said to have the inf property if for any subset A of X there is $a_0 \in A$ such that $\mu(a_0) = \inf \{ \mu(a) \mid a \in A \}$.

Definition (1.8) [11]: Let μ and σ be two fuzzy set of a set X . The Cartesian product of μ and σ denoted by $\mu \times \sigma$ is fuzzy set defined by $(\mu \times \sigma)(x, y) = \min \{ \mu(x), \sigma(y) \}$.

Definition (1.9) [8]: A fuzzy subsemigroup μ of a semigroup S is called a fuzzy bi-ideal of S if $\mu(xyz) \geq \min \{ \mu(x), \mu(z) \}$, $\forall x, y, z \in S$.

Definition (1.10) [7]: Let $M = \{x, y, z, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two non empty sets then M is called a Γ -semigroup if it satisfies

- i- $x\gamma y \in M$ ($M\Gamma M \subseteq M$), $\forall x, y \in M$ and $\gamma \in \Gamma$
- ii- $(x\beta y)\gamma z = x\beta(y\gamma z)$, $\forall x, y, z \in M$ and $\gamma, \beta \in \Gamma$.

Definition (1.11) [9]: A non empty subset A of a Γ -semigroup M is called a Γ -ideal of M if $A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

Definition (1.12) [9]: A non empty subset A of a Γ -semigroup M is called a bi- Γ -ideal of M if $A\Gamma M\Gamma A \subseteq A$.

Definition (1.13) [3]: Let M be a Γ -semigroup and M_1 a Γ_1 -semigroup. A pair of mappings $f_1: M \rightarrow M_1$ and $f_2: \Gamma \rightarrow \Gamma_1$ is said to be a homomorphism from (M, Γ) to (M_1, Γ_1) if $f_1(a\alpha b) = f_1(a)f_2(\alpha)f_1(b)$, $\forall a, b \in M, \alpha \in \Gamma$.

Definition (1.14) [3]: A fuzzy set μ of a Γ -semigroup M is called a fuzzy Γ -subsemigroup of M if $\mu(x\gamma y) \geq \min \{ \mu(x), \mu(y) \}$, $\forall x, y \in M$ and $\gamma \in \Gamma$.

Definition (1.15) [3]: A fuzzy set μ of a Γ -semigroup M is called a fuzzy Γ -ideal of M if $\mu(x\alpha y\beta z) \geq \mu(z)$ and $\mu(x\alpha y\beta z) \geq \mu(x)$, $\forall x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition (1.16) [3]: A fuzzy set μ of a Γ -semigroup M is called a fuzzy bi- Γ -ideal of M if $\forall x, y, z \in M$ and $\alpha, \gamma, \beta \in \Gamma$

- 1- $\mu(x\gamma y) \geq \min \{ \mu(x), \mu(y) \}$
- 2- $\mu(x\alpha y\beta z) \geq \min \{ \mu(x), \mu(z) \}$

Definition (1.17) [5]: Let S and S' are any two semigroups then $f: S \rightarrow S'$ is called anti-homomorphism if $f(xy) = f(y)f(x)$ $\forall x, y \in S$.

Definition (1.18) [10]: Let Q and G be any two sets. A function μ is a fuzzy set on $G \times Q$ is called a Q -fuzzy set in G .

2-The Main Results

Definition (2.1): A fuzzy set μ of a semigroup S is called an anti fuzzy subsemigroup if $\mu(xy) \leq \max \{ \mu(x), \mu(y) \}$, $\forall x, y \in S$.

Definition (2.2): An anti fuzzy subsemigroup μ of a semigroup S is called anti fuzzy bi-ideal of S if $\mu(xyz) \leq \max \{ \mu(x), \mu(z) \}$, $\forall x, y, z \in S$.

Definition (2.3): A fuzzy set μ of a Γ -semigroup is called an anti fuzzy Γ -subsemigroup of M if $\mu(x\gamma y) \leq \max \{ \mu(x), \mu(y) \}$, $\forall x, y \in M$ and $\gamma \in \Gamma$.

Definition (2.4): A fuzzy set μ of a Γ -semigroup M is called an anti fuzzy Γ -ideal of M if $\mu(x\alpha y\beta z) \leq \mu(z)$ and $\mu(x\alpha y\beta z) \leq \mu(x)$, $\forall x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition (2.5): A fuzzy set μ of a Γ -semigroup M is called anti fuzzy bi- Γ -ideal of M if

$$1-\mu(x\gamma y) \leq \max \{ \mu(x) , \mu(y) \} , \forall x,y \in M \text{ and } \gamma \in \Gamma .$$

$$2-\mu(x\alpha y\beta z) \leq \max \{ \mu(x) , \mu(z) \} , \forall x,y,z \in M \text{ and } \alpha,\beta \in \Gamma .$$

Example (2.6): For $a,b \in [0,1]$, let $M=[0,a]$ and $\Gamma=[0,b]$. Clearly M is a Γ -semigroup under usual multiplication. Moreover the fuzzy set $\mu : M \rightarrow [0,1]$ defined by

$$\mu(x) = \begin{cases} 0.5 & \text{if } x \in Q \\ 0.1 & \text{if } x \notin Q \end{cases}$$

is anti fuzzy bi- Γ -ideal of M .

Lemma (2.7): Let B be a bi- Γ -ideal of a Γ -semigroup M then for $t \in (0,1)$ there exists an anti fuzzy bi- Γ -ideal of M such that $L(\mu, t) = M$.

Proof :

Let $\mu : M \rightarrow [0,1]$ defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

where $t \in (0,1)$.

For all $x \in M$, either $x \in B$ then $\mu(x) = t$

or $x \notin B$ then $\mu(x) = 0 < t$

therefore $L(\mu, t) = M$

for all $x,y \in B$ and $\gamma \in \Gamma$ such that $x\gamma y \in B$,we have

$$\mu(x\gamma y) = t = \max \{ \mu(x) , \mu(y) \}$$

Also , $\forall x,y,z \in B$ and $\gamma,\beta \in \Gamma$ such that $x\alpha y\beta z \in B$,we have

$$\mu(x\alpha y\beta z) = t = \max \{ \mu(x) , \mu(z) \} .$$

Hence μ is anti fuzzy bi- Γ -ideal of M . ■

Lemma (2.8): Let B be a bi- Γ -ideal of a Γ -semigroup M then χ_B is an anti fuzzy bi- Γ -ideal of M .

Proof :

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

Since B is Γ -ideal of a Γ -semigroup M then $B\Gamma M \subseteq B$ and $M\Gamma B \subseteq B$

also , since B is bi- Γ -ideal of a Γ -semigroup M then $B\Gamma M\Gamma B \subseteq B$

i- Let $x,y \in M$ and $\gamma \in \Gamma$.

1- if $x,y \in B$ then $x\gamma y \in B\Gamma B \subseteq B\Gamma M \subseteq B$ and $\chi_B(x) = \chi_B(y) = \chi_B(x\gamma y) = 1$

hence $\chi_B(x\gamma y) \leq \max \{ \chi_B(x) , \chi_B(y) \}$.

2- if $x \notin B, y \in B$ then $x\gamma y \in M\Gamma B \subseteq B$ and $\chi_B(x) = 0 , \chi_B(y) = \chi_B(x\gamma y) = 1$

hence $\chi_B(x\gamma y) \leq \max \{ \chi_B(x) , \chi_B(y) \}$.

3- if $x \in B, y \notin B$ then $x\gamma y \in B\Gamma M \subseteq B$ and $\chi_B(x)=\chi_B(x\gamma y)=1$, $\chi_B(y)=0$

hence $\chi_B(x\gamma y) \leq \max \{ \chi_B(x) , \chi_B(y) \}$.

4- if $x \notin B, y \notin B$ then $x\gamma y \in M\Gamma M \not\subseteq B$ and $\chi_B(x)=\chi_B(y)=\chi_B(x\gamma y)=0$

hence $\chi_B(x\gamma y) \leq \max \{ \chi_B(x) , \chi_B(y) \}$.

ii- Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

1- if $x, z \in B$ then $x\alpha y\beta z \in B\Gamma M\Gamma B \subseteq B$ and $\chi_B(x)=\chi_B(z)=\chi_B(x\alpha y\beta z)=1$

hence $\chi_B(x\alpha y\beta z) \leq \max \{ \chi_B(x) , \chi_B(z) \}$.

2- if $x \notin B, z \in B$ then $x\alpha y\beta z \in M\Gamma M\Gamma B \not\subseteq B$ and $\chi_B(x)=\chi_B(x\alpha y\beta z)=0$, $\chi_B(z)=1$

hence $\chi_B(x\alpha y\beta z) \leq \max \{ \chi_B(x) , \chi_B(z) \}$.

3- if $x \in B, z \notin B$ then $x\alpha y\beta z \in B\Gamma M\Gamma M \not\subseteq B$ and $\chi_B(x)=1$, $\chi_B(z)=\chi_B(x\alpha y\beta z)=0$

hence $\chi_B(x\alpha y\beta z) \leq \max \{ \chi_B(x) , \chi_B(z) \}$.

4- if $x \notin B, z \notin B$ then $x\alpha y\beta z \in M\Gamma M\Gamma M \not\subseteq B$ and $\chi_B(x)=\chi_B(z)=\chi_B(x\alpha y\beta z)=0$

hence $\chi_B(x\alpha y\beta z) \leq \max \{ \chi_B(x) , \chi_B(z) \}$.

Thus, χ_B is anti fuzzy bi- Γ -ideal of M . ■

Theorem (2.9): If $\{\mu_j : j \in \Lambda\}$ is a family of anti fuzzy bi- Γ -ideal of a Γ -semigroup M then $\bigcup_{j \in \Lambda} \mu_j$ is an anti fuzzy bi- Γ -ideal of M .

Proof : $\bigcup_{j \in \Lambda} \mu_j = \sup_{j \in \Lambda} \{\mu_j(x) : x \in M\}$

i- Let $x, y \in B$ and $\gamma \in \Gamma$,

$$\begin{aligned} \bigcup_{j \in \Lambda} \mu_j(x\gamma y) &= \sup_{j \in \Lambda} \{\mu_j(x\gamma y)\} \\ &\leq \sup_{j \in \Lambda} \{\max\{\mu_j(x), \mu_j(y)\}\} \\ &= \max\{\sup_{j \in \Lambda} (\mu_j(x)), \sup_{j \in \Lambda} (\mu_j(y))\} \\ &= \max\{\{\sup_{j \in \Lambda} (\mu_j(x)) : x \in M\}, \{\sup_{j \in \Lambda} (\mu_j(y)) : y \in M\}\} \\ &= \max\{\bigcup_{j \in \Lambda} \mu_j(x), \bigcup_{j \in \Lambda} \mu_j(y)\} \end{aligned}$$

ii- Let $x, y, z \in B$ and $\alpha, \beta \in \Gamma$

$$\begin{aligned} \bigcup_{j \in \Lambda} \mu_j(x\alpha y\beta z) &= \sup_{j \in \Lambda} \{\mu_j(x\alpha y\beta z)\} \\ &\leq \sup_{j \in \Lambda} \{\max\{\mu_j(x), \mu_j(z)\}\} \\ &= \max\{\sup_{j \in \Lambda} (\mu_j(x)), \sup_{j \in \Lambda} (\mu_j(z))\} \\ &= \max\{\{\sup_{j \in \Lambda} (\mu_j(x)) : x \in M\}, \{\sup_{j \in \Lambda} (\mu_j(z)) : z \in M\}\} \\ &= \max\{\bigcup_{j \in \Lambda} \mu_j(x), \bigcup_{j \in \Lambda} \mu_j(z)\} \end{aligned}$$

Hence $\bigcup_{j \in \Lambda} \mu_j$ is an anti fuzzy bi- Γ -ideal of M . ■

Theorem (2.10): Let μ be a fuzzy set of a Γ -semigroup M then μ is a fuzzy bi- Γ -ideal of M if and only if μ^c is an anti fuzzy bi- Γ -ideal of M .

Proof :

suppose μ is fuzzy bi- Γ -ideal of M

i- Let $x, y \in M$ and $\gamma \in \Gamma$,

$$\mu(x\gamma y) \geq \min \{ \mu(x), \mu(y) \}$$

$$1 - \mu^c(x\gamma y) \geq \min \{ 1 - \mu^c(x), 1 - \mu^c(y) \}$$

$$\mu^c(x\gamma y) \leq 1 - \min \{ 1 - \mu^c(x), 1 - \mu^c(y) \}$$

$$\mu^c(x\gamma y) \geq \max \{ \mu^c(x), \mu^c(y) \}$$

ii- Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

$$\mu(x\alpha y\beta z) \geq \min \{ \mu(x), \mu(z) \}$$

$$1 - \mu^c(x\alpha y\beta z) \geq \min \{ 1 - \mu^c(x), 1 - \mu^c(z) \}$$

$$\mu^c(x\alpha y\beta z) \leq 1 - \min \{ 1 - \mu^c(x), 1 - \mu^c(z) \}$$

$$\mu^c(x\alpha y\beta z) \leq \max \{ \mu^c(x), \mu^c(z) \}$$

Hence μ^c is an anti fuzzy bi- Γ -ideal of M .

Conversely suppose μ^c is an anti fuzzy bi- Γ -ideal of M .

i- Let $x, y \in M$ and $\gamma \in \Gamma$,

$$\mu^c(x\gamma y) \geq \max \{ \mu^c(x), \mu^c(y) \}$$

$$1 - \mu(x\gamma y) \geq \max \{ 1 - \mu(x), 1 - \mu(y) \}$$

$$\mu(x\gamma y) \leq 1 - \max \{ 1 - \mu(x), 1 - \mu(y) \}$$

$$\mu(x\gamma y) \leq \min \{ \mu(x), \mu(y) \}$$

ii- Let $\forall x, y, z \in M$ and $\alpha, \beta \in \Gamma$

$$\mu^c(x\alpha y\beta z) \geq \max \{ \mu^c(x), \mu^c(z) \}$$

$$1 - \mu(x\alpha y\beta z) \geq \max \{ 1 - \mu(x), 1 - \mu(z) \}$$

$$\mu(x\alpha y\beta z) \leq 1 - \max \{ 1 - \mu(x), 1 - \mu(z) \}$$

$$\mu(x\alpha y\beta z) \leq \min \{ \mu(x), \mu(z) \}$$

Hence μ is a fuzzy bi- Γ -ideal of M . ■

Theorem (2.11): A fuzzy set μ in a Γ -ideal of a Γ -semigroup M is an anti fuzzy bi- Γ -ideal of M if and only if the lower level set $L(\mu, t)$ is a bi- Γ -ideal in M when it is non empty.

Proof :

Suppose μ is an anti fuzzy bi- Γ -ideal of M

i- Let $x, y \in L(\mu, t)$ and $\gamma \in \Gamma$,

$$\mu(x) \leq t, \mu(y) \leq t \text{ then}$$

$$\mu(x\gamma y) \leq \max \{ \mu(x), \mu(y) \} \leq t$$

$x\gamma y \in L(\mu, t)$, therefore
 $L(\mu, t) \Gamma L(\mu, t) \subseteq L(\mu, t)$
Hence $L(\mu, t)$ is a sub Γ - semigroup
ii- Let $x, z \in L(\mu, t)$, $y \in M$ and $\alpha, \beta \in \Gamma$
 $\mu(x) \leq t$ and $\mu(z) \leq t$
 $\mu(x\alpha y\beta z) \leq \max \{ \mu(x) , \mu(z) \} \leq t$
 $x\alpha y\beta z \in L(\mu, t)$

Hence, $L(\mu, t)$ is a bi- Γ - ideal in M

Conversely, suppose $L(\mu, t)$ is a bi- Γ - ideal in M

i- Let $x, y \in M$ and $\gamma \in \Gamma$, and $t = \max \{ \mu(x) , \mu(y) \}$ then
 $x, y \in L(\mu, t)$ and $x\gamma y \in L(\mu, t)$, $\mu(x\gamma y) \leq t = \max \{ \mu(x) , \mu(y) \}$
ii- Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ and $t = \max \{ \mu(x) , \mu(z) \}$ then
 $x, y, z \in L(\mu, t)$, $x\alpha y\beta z \in L(\mu, t)$
 $\mu(x\alpha y\beta z) \leq t = \max \{ \mu(x) , \mu(z) \}$

Therefore $L(\mu, t) \Gamma M \Gamma L(\mu, t) \subseteq L(\mu, t)$

hence , μ is an anti fuzzy bi- Γ -ideal of M . ■

Theorem (2.12): Let M_1 be Γ_1 -semigroup and M_2 be Γ_2 -semigroup and the pair of mapping $f: M_1 \rightarrow M_2$, $h: \Gamma_1 \rightarrow \Gamma_2$ be a homomorphism if μ is an anti fuzzy bi- Γ_2 -ideal of M_2 , then $f^{-1}(\mu)$ is an anti fuzzy bi- Γ_1 -ideal of M_1 .

Proof :

i- Let $x, y \in M_1$ and $\gamma \in \Gamma_1$,
 $f^{-1}(\mu)(x\gamma y) = \mu(f(x\gamma y))$
 $= \mu(f(x)h(\gamma)f(y))$
 $\leq \max \{ \mu(f(x)) , \mu(f(y)) \}$
 $= \max \{ f^{-1}(\mu)(x) , f^{-1}(\mu)(y) \}$
ii- Let $x, y, z \in M_1$ and $\alpha, \beta \in \Gamma_1$
 $f^{-1}(\mu)(x\alpha y\beta z) = \mu(f(x\alpha y\beta z))$
 $= \mu(f(x)h(\alpha)f(y)h(\beta)f(z))$
 $\leq \max \{ \mu(f(x)) , \mu(f(z)) \}$
 $= \max \{ f^{-1}(\mu)(x) , f^{-1}(\mu)(z) \}$

Hence $f^{-1}(\mu)$ is an anti fuzzy bi- Γ_1 -ideal of M_1 . ■

Definition (2.13): Let M_1 be Γ_1 -semigroup and M_2 be Γ_2 -semigroup . The pair of mapping $f: M_1 \rightarrow M_2$, $h: \Gamma_1 \rightarrow \Gamma_2$ is said to be an anti homomorphism from (M_1, Γ_1) to (M_2, Γ_2) if $f(x\gamma y) = f(y)h(\gamma)f(x) \forall x, y \in M_1$ and $\gamma \in \Gamma_1$.

Theorem (2.14): Let M_1 be Γ_1 -semigroup and M_2 be Γ_2 -semigroup and the pair of mapping $f: M_1 \rightarrow M_2$, $h: \Gamma_1 \rightarrow \Gamma_2$ be an anti homomorphism if μ is an anti fuzzy bi Γ_2 -ideal of M_2 , then $f^{-1}(\mu)$ is an anti fuzzy bi- Γ_1 -ideal of M_1 .

Proof :

i- Let $x, y \in M_1$ and $\gamma \in \Gamma_1$,

$$\begin{aligned} f^{-1}(\mu)(x\gamma y) &= \mu(f(x\gamma y)) \\ &= \mu(f(y)h(\gamma)f(x)) \\ &\leq \max\{\mu(f(y)), \mu(f(x))\} \\ &= \max\{f^{-1}(\mu)(y), f^{-1}(\mu)(x)\} \end{aligned}$$

ii- Let $x, y, z \in M_1$ and $\alpha, \beta \in \Gamma_1$

$$\begin{aligned} f^{-1}(\mu)(x\alpha y\beta z) &= \mu(f(x\alpha y\beta z)) \\ &= \mu(f(z)h(\alpha)f(y)h(\beta)f(x)) \\ &\leq \max\{\mu(f(z)), \mu(f(x))\} \\ &= \max\{f^{-1}(\mu)(z), f^{-1}(\mu)(x)\} \end{aligned}$$

Hence $f^{-1}(\mu)$ is an anti fuzzy bi- Γ_1 -ideal of M_1 . ■

Theorem (2.15): Let M_1 be Γ_1 -semigroup and M_2 be Γ_2 -semigroup and the pair of mapping $f: M_1 \rightarrow M_2$, $h: \Gamma_1 \rightarrow \Gamma_2$ be a homomorphism if μ is an anti fuzzy bi- Γ_2 -ideal of M_2 and $L(\mu, t)$ is bi- Γ_2 -ideal in M_2 , then $L(f^{-1}(\mu), t)$ is bi- Γ_1 -ideal in M_1

Proof :

i- Let $x, y \in L(f^{-1}(\mu), t)$ and $\gamma \in \Gamma_1$,

$$\begin{aligned} f^{-1}(\mu)(x) &= \mu(f(x)) \leq t, \quad f^{-1}(\mu)(y) = \mu(f(y)) \leq t \\ f^{-1}(\mu)(x\gamma y) &= \mu(f(x\gamma y)) \\ &= \mu(f(y)h(\gamma)f(x)) \\ &\leq \max\{\mu(f(y)), \mu(f(x))\} \leq t \end{aligned}$$

$$x\gamma y \in L(f^{-1}(\mu), t)$$

ii- Let $x, y, z \in L(f^{-1}(\mu), t)$ and $\alpha, \beta \in \Gamma_1$,

$$\begin{aligned} f^{-1}(\mu)(x) &= \mu(f(x)) \leq t, \quad f^{-1}(\mu)(z) = \mu(f(z)) \leq t \\ f^{-1}(\mu)(x\alpha y\beta z) &= \mu(f(x\alpha y\beta z)) \\ &= \mu(f(z)h(\alpha)f(y)h(\beta)f(x)) \\ &\leq \max\{\mu(f(x)), \mu(f(z))\} \leq t \end{aligned}$$

$$x\alpha y\beta z \in L(f^{-1}(\mu), t)$$

Hence, $L(f^{-1}(\mu), t)$ is bi- Γ_1 -ideal in M_1 . ■

We can prove the following theorem in a similar manner

Theorem (2.16): Let M_1 be Γ_1 -semigroup and M_2 be Γ_2 -semigroup and the pair of mapping $f: M_1 \rightarrow M_2$, $h: \Gamma_1 \rightarrow \Gamma_2$ be an anti homomorphism if μ is an anti fuzzy bi- Γ_2 - ideal of M_2 and $L(\mu, t)$ is bi- Γ_2 - ideal in M_2 , then $L(f^{-1}(\mu), t)$ is bi- Γ_1 - ideal in M_1 .

Theorem (2.17): Let M_1 be Γ_1 -semigroup and M_2 be Γ_2 -semigroup and the pair of mapping $f: M_1 \rightarrow M_2$, $h: \Gamma_1 \rightarrow \Gamma_2$ be a homomorphism if μ is an anti fuzzy bi- Γ_1 - ideal of M_1 which has the inf property and $L(\mu, t)$ is bi- Γ_1 - ideal in M_1 , then $L(f(\mu), t)$ is bi- Γ_2 - ideal in $f(M_1)$.

Proof :

Let $f(x), f(y), f(z) \in L(f(\mu), t)$

$(f(\mu))(f(x)) \leq t, (f(\mu))(f(y)) \leq t, (f(\mu))(f(z)) \leq t$

let $x_0 \in f^{-1}(f(x)), y_0 \in f^{-1}(f(y)), z_0 \in f^{-1}(f(z))$, be such that

$$\mu(x_0) = \inf_{g \in f^{-1}(f(x))} \mu(g), \mu(y_0) = \inf_{g \in f^{-1}(f(y))} \mu(g), \mu(z_0) = \inf_{g \in f^{-1}(f(z))} \mu(g)$$

i- Let $\gamma \in \Gamma_2$

$$(f(\mu))(f(x)\gamma f(y)) = \inf_{w \in f^{-1}(f(x)\gamma f(y))} \mu(w)$$

$$\leq \max \{ \mu(x_0), \mu(y_0) \}$$

$$= \max \{ (f(\mu))(f(x)), (f(\mu))(f(y)) \} \leq t$$

ii- Let $\alpha, \beta \in \Gamma_2$

$$(f(\mu))(f(x)\alpha f(y)\beta f(z)) = \inf_{w \in f^{-1}(f(x)\alpha f(y)\beta f(z))} \mu(w)$$

$$\leq \max \{ \mu(x_0), \mu(z_0) \}$$

$$= \max \{ (f(\mu))(f(x)), (f(\mu))(f(z)) \} \leq t$$

Hence, $L(f(\mu), t)$ is bi- Γ_2 -ideal of $f(\mu)$. ■

We can prove the following theorem in a similar manner

Theorem (2.18): Let M_1 be Γ_1 -semigroup and M_2 be Γ_2 -semigroup and the pair of mapping $f: M_1 \rightarrow M_2$, $h: \Gamma_1 \rightarrow \Gamma_2$ be an anti homomorphism if μ is an anti fuzzy bi- Γ_1 - ideal of M_1 which has the inf property and $L(\mu, t)$ is bi- Γ_1 - ideal in M_1 , then $L(f(\mu), t)$ is bi- Γ_2 - ideal in $f(M_1)$.

Definition (2.19): A fuzzy set μ is called anti fuzzy interior Γ -ideal of a Γ -semigroup M if

$$\mu(x\alpha y\beta z) \leq \mu(y) \quad \forall x, y, z \in M \text{ and } \alpha, \beta \in \Gamma.$$

Theorem (2.20): Let A be an interior of a Γ -semigroup M then τ_A is anti fuzzy interior Γ - ideal of M , where τ_A defined by

$$\tau_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

Proof :

Let $x, y, z \in A$ and $\alpha, \beta \in \Gamma$ then $x\alpha y\beta z \in M\Gamma A\Gamma M \subseteq A$

i- if $y \in A$ then $\tau_A(y) = 0$

Thus , $\tau_A(x\alpha y\beta z) \leq \tau_A(y)$

ii- if $y \notin A$ then $\tau_A(y) = 1$

Thus , $\tau_A(x\alpha y\beta z) \leq \tau_A(y)$

Hence , τ_A is anti fuzzy interior Γ - ideal of M . ■

Definition (2.21): Let μ and σ be two fuzzy sets of a nonempty set S then $(\mu \times \sigma)(x, y) = \max \{ \mu(x), \sigma(y) \} \forall x, y \in S$ is called anti cartesian product of μ and σ .

Theorem (2.22): If μ and σ are anti fuzzy bi-ideal of a semigroups S_1 and S_2 respectively , then $\mu \times \sigma$ is anti fuzzy bi-ideal of $S_1 \times S_2$.

Proof :

Let $(a, b), (c, d)$ and $(e, f) \in S_1 \times S_2$

$$\begin{aligned} (\mu \times \sigma)((a, b)(c, d)(e, f)) &= (\mu \times \sigma)(ace, bdf) \\ &= \max \{ \mu(ace), \sigma(bdf) \} \\ &\leq \max \{ \max \{ \mu(a), \mu(e) \}, \max \{ \sigma(b), \sigma(f) \} \} \\ &\leq \max \{ \max \{ \mu(a), \sigma(b) \}, \max \{ \mu(e), \sigma(f) \} \} \\ &= \max \{ (\mu \times \sigma)(a, b), (\mu \times \sigma)(e, f) \} \end{aligned}$$

Hence $\mu \times \sigma$ is anti fuzzy bi-ideal of $S_1 \times S_2$. ■

Theorem (2.23): Let μ is an anti fuzzy bi- Γ_1 -ideal of a Γ -semigroup M_1 and σ is an anti fuzzy bi- Γ_2 -ideal of a Γ -semigroup M_2 then $\mu \times \sigma$ is anti fuzzy bi- $\Gamma_1 \times \Gamma_2$ ideal of $M_1 \times M_2$

Proof :

i- Let $(a, b), (c, d) \in M_1 \times M_2$ and $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$

$$\begin{aligned} (\mu \times \sigma)((a, b)(\gamma_1, \gamma_2)(c, d)) &= (\mu \times \sigma)(a\gamma_1 c, b\gamma_2 d) \\ &= \max \{ \mu(a\gamma_1 c), \sigma(b\gamma_2 d) \} \\ &\leq \max \{ \max \{ \mu(a), \mu(c) \}, \max \{ \sigma(b), \sigma(d) \} \} \\ &\leq \max \{ \max \{ \mu(a), \sigma(b) \}, \max \{ \mu(c), \sigma(d) \} \} \\ &= \max \{ (\mu \times \sigma)(a, b), (\mu \times \sigma)(c, d) \} \end{aligned}$$

$$\begin{aligned}
\text{ii- Let } (a,b), (c,d), (e,f) \in M_1 \times M_2 \text{ and } (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Gamma_1 \times \Gamma_2 \\
(\mu \times \sigma)((a,b) (\alpha_1, \alpha_2)(c,d)(\beta_1, \beta_2) (e,f)) &= (\mu \times \sigma)(a\alpha_1 c\beta_1 e, b\alpha_2 d\beta_2 f) \\
&= \max \{ \mu(a\alpha_1 c\beta_1 e), \sigma(b\alpha_2 d\beta_2 f) \} \\
&\leq \max \{ \max \{ \mu(a), \mu(e) \}, \max \{ \sigma(b), \sigma(f) \} \} \\
&\leq \max \{ \max \{ \mu(a), \sigma(b) \}, \max \{ \mu(e), \sigma(f) \} \} \\
&= \max \{ (\mu \times \sigma)(a,b), (\mu \times \sigma)(e,f) \}
\end{aligned}$$

Hence $\mu \times \sigma$ is anti fuzzy bi- $\Gamma_1 \times \Gamma_2$ -ideal of $M_1 \times M_2$. ■

Theorem (2.24): Let μ, σ be fuzzy subset of a non empty set M and $\mu \times \sigma$ is an anti fuzzy bi-ideal of $M \times M$ such that either $\mu(a) \leq \sigma(b)$ or $\sigma(b) \leq \mu(a) \forall a, b \in M$ then either μ is anti fuzzy bi- ideal of M or σ anti fuzzy bi-ideal of M

Proof :

$$\begin{aligned}
(\mu \times \sigma)(x,y) &= \max \{ \mu(x), \sigma(y) \} \text{ and} \\
(\mu \times \sigma)((x,y)(w,z)) &= (\mu \times \sigma)(xw, yz) = \max \{ \mu(xw), \sigma(yz) \} \dots\dots\dots(1) \\
\text{also}
\end{aligned}$$

$$\begin{aligned}
(\mu \times \sigma)((x,y)(w,z)) &\leq \max \{ (\mu \times \sigma)(x,y), (\mu \times \sigma)(w,z) \} \\
&= \max \{ \max \{ \mu(x), \sigma(y) \}, \max \{ \mu(w), \sigma(z) \} \} \dots\dots\dots(2)
\end{aligned}$$

From (1) and (2) ,we obtain

$$\max \{ \mu(xw), \sigma(yz) \} \leq \max \{ \max \{ \mu(x), \sigma(y) \}, \max \{ \mu(w), \sigma(z) \} \} \dots\dots\dots(3)$$

i- Suppose $\mu(a) \leq \sigma(b) \forall a, b \in M$

From (3) we obtain

$$\sigma(yz) \leq \max \{ \sigma(y), \sigma(z) \} \forall y, z \in M$$

Hence σ anti fuzzy bi-ideal of M

ii- Suppose $\sigma(b) \leq \mu(a) \forall a, b \in M$

From (3) we obtain

$$\mu(xw) \leq \max \{ \mu(x), \sigma(w) \} \forall x, w \in M$$

Hence μ anti fuzzy bi-ideal of M . ■

Theorem (2.25): Let μ and σ be fuzzy subset of a non empty set M and $\mu \times \sigma$ is an anti fuzzy bi- Γ - ideal of $M \times M$ such that either $\mu(a) \leq \sigma(b)$ or $\sigma(b) \leq \mu(a) \forall a, b \in M$ then either μ is anti fuzzy bi- Γ - ideal of M or σ anti fuzzy bi- Γ -ideal of M

Proof :

$$\begin{aligned}
(\mu \times \sigma)(x,y) &= \max \{ \mu(x), \sigma(y) \} \text{ and} \\
(\mu \times \sigma)((x,y)(\gamma_1, \gamma_2)(w,z)) &= (\mu \times \sigma)(x\gamma_1 w, y\gamma_2 z) \\
&= \max \{ \mu(x\gamma_1 w), \sigma(y\gamma_2 z) \} \dots\dots\dots(1)
\end{aligned}$$

also

$$\begin{aligned}
(\mu \times \sigma)((x,y)(\gamma_1,\gamma_2)(w,z)) &\leq \max \{(\mu \times \sigma)(x,y), (\mu \times \sigma)(w,z)\} \\
&= \max \{ \max \{ \mu(x), \sigma(y) \}, \max \{ \mu(w), \sigma(z) \} \} \dots\dots\dots(2)
\end{aligned}$$

From (1) and (2) ,we obtain

$$\max \{ \mu(x\gamma_1w), \sigma(y\gamma_2z) \} \leq \max \{ \max \{ \mu(x), \sigma(y) \}, \max \{ \mu(w), \sigma(z) \} \} \dots\dots\dots(3)$$

$$\begin{aligned}
(\mu \times \sigma)((x,y)(\alpha_1,\alpha_2)(w,z)(\beta_1,\beta_2)(g,h)) &= (\mu \times \sigma)(x\alpha_1w\beta_1g, y\alpha_2z\beta_2h) \\
&= \max \{ \mu(x\alpha_1w\beta_1g), \sigma(y\alpha_2z\beta_2h) \} \dots\dots\dots(4)
\end{aligned}$$

also

$$\begin{aligned}
(\mu \times \sigma)((x,y)(\alpha_1,\alpha_2)(w,z)(\beta_1,\beta_2)(g,h)) &\leq \max \{(\mu \times \sigma)(x,y), (\mu \times \sigma)(g,h)\} \\
&= \max \{ \max \{ \mu(x), \sigma(y) \}, \max \{ \mu(g), \sigma(h) \} \} \dots\dots\dots(5)
\end{aligned}$$

From (4) and (5) ,we obtain

$$\max \{ \mu(x\alpha_1w\beta_1g), \sigma(y\alpha_2z\beta_2h) \} \leq \max \{ \max \{ \mu(x), \sigma(y) \}, \max \{ \mu(g), \sigma(h) \} \} \dots\dots\dots(6)$$

Because M is a Γ -semigroup then $x\alpha_1w, y\alpha_2z \in M$ and $x\alpha_1w\beta_1g, y\alpha_2z\beta_2h \in M$

i- Suppose $\mu(a) \leq \sigma(b) \forall a,b \in M$

From (3) we obtain

$$\sigma(y\gamma_2z) \leq \max \{ \sigma(y), \sigma(z) \} \forall y,z \in M, \gamma \in \Gamma$$

Also from (6), we obtain

$$\sigma(y\alpha_2z\beta_2h) \leq \max \{ \sigma(y), \sigma(h) \} \forall y,z,h \in M, \alpha_2, \beta_2 \in \Gamma$$

Hence σ anti fuzzy bi- Γ -ideal of M

ii- Suppose $\sigma(b) \leq \mu(a) \forall a,b \in M$

From (3) we obtain

$$\mu(x\gamma_1w) \leq \max \{ \mu(x), \mu(w) \} \forall x,w \in M, \gamma_1 \in \Gamma$$

Also from (6), we obtain

$$\mu(x\alpha_1w\beta_1g) \leq \max \{ \max \{ \mu(x), \mu(g) \} \} \forall x,w,g \in M, \alpha_1, \beta_1 \in \Gamma$$

Hence μ anti fuzzy bi- Γ -ideal of M . ■

Definition (2.26): A Q-fuzzy set μ of a semigroup S is called an anti Q-fuzzy subsemigroup if $\mu(xy,q) \leq \max \{ \mu(x,q), \mu(y,q) \}, \forall x,y \in S, q \in Q$.

Definition (2.27): An anti Q-fuzzy subsemigroup μ of a semigroup S is called anti Q-fuzzy bi-ideal of S if $\mu(xyz,q) \leq \max \{ \mu(x,q), \mu(z,q) \}, \forall x,y,z \in S, q \in Q$.

Definition (2.28) : A Q-fuzzy set μ of a Γ -semigroup is called an anti Q-fuzzy Γ -subsemigroup of M if $\mu(x\gamma y,q) \leq \max \{ \mu(x,q), \mu(y,q) \}, \forall x,y \in M, q \in Q$ and $\gamma \in \Gamma$.

Definition (2.29): A Q-fuzzy set μ of a Γ -semigroup M is called anti Q-fuzzy Γ -ideal of M if $\mu(x\alpha y\beta z,q) \leq \mu(z,q)$ and $\mu(x\alpha y\beta z,q) \leq \mu(x,q), \forall x,y,z \in M, q \in Q$ and $\alpha, \beta \in \Gamma$. [3]

Definition (2.30): A Q-fuzzy set μ of a Γ -semigroup M is called anti Q-fuzzy bi- Γ -ideal of M if

$$1-\mu(x\gamma y, q) \leq \max \{ \mu(x, q) , \mu(y, q) \} , \forall x, y \in M, q \in Q \text{ and } \gamma \in \Gamma .$$

$$2-\mu(x\alpha y\beta z, q) \leq \max \{ \mu(x, q) , \mu(z, q) \} , \forall x, y, z \in M, q \in Q \text{ and } \alpha, \beta \in \Gamma .$$

Theorem (2.31): Let μ be Q-fuzzy set of a Γ -semigroup M then μ is Q-fuzzy bi- Γ -ideal of M if and only if μ^c is an anti Q-fuzzy bi- Γ -ideal of M .

Proof :

suppose μ is Q-fuzzy bi- Γ -ideal of M

i- Let $x, y \in M, q \in Q$ and $\gamma \in \Gamma$,

$$\mu(x\gamma y, q) \geq \min \{ \mu(x, q) , \mu(y, q) \}$$

$$1-\mu^c(x\gamma y, q) \geq \min \{ 1-\mu^c(x, q) , 1-\mu^c(y, q) \}$$

$$\mu^c(x\gamma y, q) \leq 1-\min \{ 1-\mu^c(x, q) , 1-\mu^c(y, q) \}$$

$$\mu^c(x\gamma y, q) \geq \max \{ \mu^c(x, q) , \mu^c(y, q) \}$$

ii-Let $x, y, z \in M, q \in Q$ and $\alpha, \beta \in \Gamma$

$$\mu(x\alpha y\beta z, q) \geq \min \{ \mu(x, q) , \mu(z, q) \}$$

$$1-\mu^c(x\alpha y\beta z, q) \geq \min \{ 1-\mu^c(x, q) , 1-\mu^c(z, q) \}$$

$$\mu^c(x\alpha y\beta z, q) \leq 1-\min \{ 1-\mu^c(x, q) , 1-\mu^c(z, q) \}$$

$$\mu^c(x\alpha y\beta z, q) \leq \max \{ \mu^c(x, q) , \mu^c(z, q) \}$$

Hence μ^c is an anti fuzzy bi- Γ -ideal of M .

Conversely suppose μ^c is an anti fuzzy bi- Γ -ideal of M .

i- Let $x, y \in M, q \in Q$ and $\gamma \in \Gamma$,

$$\mu^c(x\gamma y, q) \geq \max \{ \mu^c(x, q) , \mu^c(y, q) \}$$

$$1-\mu(x\gamma y, q) \geq \max \{ 1-\mu(x, q) , 1-\mu(y, q) \}$$

$$\mu(x\gamma y, q) \leq 1-\max \{ 1-\mu(x, q) , 1-\mu(y, q) \}$$

$$\mu(x\gamma y, q) \leq \min \{ \mu(x, q) , \mu(y, q) \}$$

ii-Let $\forall x, y, z \in M, q \in Q$ and $\alpha, \beta \in \Gamma$

$$\mu^c(x\alpha y\beta z, q) \geq \max \{ \mu^c(x, q) , \mu^c(z, q) \}$$

$$1-\mu(x\alpha y\beta z, q) \geq \max \{ 1-\mu(x, q) , 1-\mu(z, q) \}$$

$$\mu(x\alpha y\beta z, q) \leq 1-\max \{ 1-\mu(x, q) , 1-\mu(z, q) \}$$

$$\mu(x\alpha y\beta z, q) \leq \min \{ \mu(x, q) , \mu(z, q) \}$$

Hence μ is a Q-fuzzy bi- Γ -ideal of M . ■

Theorem (2.32): A Q-fuzzy set μ in a Γ -ideal of a Γ -semigroup M is an anti Q-fuzzy bi- Γ -ideal of M if and only if the lower level set $L(\mu, t)$ is a bi- Γ -ideal in M when it is non empty .

Proof :

Suppose μ is an Q-anti fuzzy bi- Γ -ideal of M

i- Let $x, y \in L(\mu, t), q \in Q$ and $\gamma \in \Gamma$,

$$\mu(x, q) \leq t , \mu(y, q) \leq t$$

$\mu(x\gamma y, q) \leq \max \{ \mu(x, q) , \mu(y, q) \} \leq t$
 $x\gamma y \in L(\mu, t)$
 ii- Let $x, y, z \in L(\mu, t)$, $q \in Q$ and $\alpha, \beta \in \Gamma$
 $\mu(x, q) \leq t$, $\mu(y, q) \leq t$ and $\mu(z, q) \leq t$
 $\mu(x\alpha y\beta z, q) \leq \max \{ \mu(x, q) , \mu(z, q) \} \leq t$
 $x\alpha y\beta z \in L(\mu, t)$

Hence, $L(\mu, t)$ is a bi- Γ - ideal in M

Conversely, suppose $L(\mu, t)$ is a bi- Γ - ideal in M

i- Let $x, y \in M$, $q \in Q$ and $\gamma \in \Gamma$, and $t = \max \{ \mu(x, q) , \mu(y, q) \}$ then
 $x, y \in L(\mu, t)$ and $x\gamma y \in L(\mu, t)$, $\mu(x\gamma y, q) \leq t = \max \{ \mu(x, q) , \mu(y, q) \}$
 ii- Let $x, y, z \in M$, $q \in Q$ and $\alpha, \beta \in \Gamma$ and $t = \max \{ \mu(x, q) , \mu(z, q) \}$ then
 $x, y, z \in L(\mu, t)$, $x\alpha y\beta z \in L(\mu, t)$
 $\mu(x\alpha y\beta z, q) \leq t = \max \{ \mu(x, q) , \mu(z, q) \}$
 hence , μ is an anti Q-fuzzy bi- Γ -ideal of M . ■

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ضد مثاليات كاما الثنائية الضبابية في شبه الزمرة كاما

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المستخلص

في هذا البحث عرفنا البنى الجبرية ضد مثاليات كاما الثنائية الضبابية و ضد مثاليات كاما الثنائية الضبابية من النوع Q ودراسة بعض الخواص الأساسية لهما