

The Composition Operator C_{ϕ_λ} Induced by The Function ϕ_λ

By

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Abstract

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions $f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n)z^n$ holomorphic on U such that $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ with $f^{\wedge}(n)$ denotes then the Taylor coefficient of f .

Let ψ be a holomorphic self-map of U , the composition operator C_ψ induced by ψ is defined on H^2 by the equation

$$C_\psi f = f \circ \psi \quad (f \in H^2)$$

We have studied the composition operator induced by the automorphism ϕ_λ and discussed the adjoint of the composition of the symbol ϕ_λ . We have look also at some known properties on composition operators and tried to see the analogue properties in order to show how the results are changed by changing the function ψ in U .

In order to make the work accessible to the reader, we have included some known results with the details of the proofs for some cases and proofs for the properties.

Introduction

This search consists of two sections. In section one, we are going to the automorphism ϕ_λ and properties of ϕ_λ , and also discuss the interior and exterior fixed points of ϕ_λ and also discuss ϕ_λ is rotation a round the origin and ϕ_λ is elliptic and ϕ_λ is a linear fractional transformation.

In section two, we are going to the Composition Operator C_{ϕ_λ} induced by the symbol ϕ_λ and properties of C_{ϕ_λ} , and also discuss the adjoint of Composition Operator C_{ϕ_λ} induced by the symbol ϕ_λ and also discuss C_{ϕ_λ} is an invertible operator and C_{ϕ_λ} is normal operator and define eigenvalue of C_{ϕ_λ} .

Section One

Definition(1.1) : [1]

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ is called unit ball in complex $\mathbb{C}$ and $\partial U = \{z \in \mathbb{C} : |z| = 1\}$ is called boundary of U

Example(1.2):

For $\lambda \in U$, define $\phi_\lambda(z) = \frac{z}{\lambda z - 1}$ ($z \in U$). Since the denominator equal zero only at $z = \frac{1}{\lambda}$, the function ϕ_λ is holomorphic on the ball $\{z \in \mathbb{C} : |z| < \frac{1}{|\lambda|}\}$. Since $\lambda \in U$, then this ball contains U . Hence ϕ_λ take U into U and holomorphic on U .

Definition(1.3) : [1]

Let $\psi : U \rightarrow U$ and holomorphic on U . We say that ψ is called conformal automorphism or automorphism of U if and only if ψ is injective and surjective.

Proposition (1.4) :

for $\lambda \in U$, ϕ_λ is conformal automorphism or automorphism of U .

Proof:

Since $\phi_\lambda(z) = \frac{z}{\lambda z - 1}$ ($z, \lambda \in U$)

Suppose $\phi_\lambda(z_1) = \phi_\lambda(z_2)$ that is $\frac{z_1}{\lambda z_1 - 1} = \frac{z_2}{\lambda z_2 - 1}$, therefore $\lambda z_1 z_2 - z_1 = \lambda z_1 z_2 - z_2$, hence $z_1 = z_2$. Thus

ϕ_λ is injective.

Let $y = \phi_\lambda(z)$, that is $y = \frac{z}{\lambda z - 1}$, therefore $\lambda z y - y = z$, then $\lambda z y - z = y$, hence

$$z = \frac{y}{\lambda y - 1}, \quad \phi_\lambda(z) = \phi_\lambda\left(\frac{y}{\lambda y - 1}\right) = \frac{\frac{y}{\lambda y - 1}}{\frac{\lambda y}{\lambda y - 1} - 1} = \frac{\frac{y}{\lambda y - 1}}{\frac{\lambda y - 1}{\lambda y - 1}} = y, \text{ for every } y \in U \text{ there exists}$$

$z \in U$ such that $\phi_\lambda(z) = y$. Thus ϕ_λ is surjective. Hence ϕ_λ is automorphism.

Definition(1.9) : [1.9]

A point $p \in C$ is a fixed point for the the function ψ , if $\psi(p) = p$.

Proposition (1.10) :

For $\lambda \in U$, then $0, \frac{2}{\lambda}$ are fixed points for ϕ_λ .

Proof :

Let $\phi_\lambda(z) = z$ that is $\frac{z}{\lambda z - 1} = z$, therefore $\lambda z^2 - 2z = 0$. Hence ϕ_λ has two fixed points

$$z_1 = 0, z_2 = \frac{2}{\lambda}$$

Definition(1.11): [1.11]

Let $\psi : U \rightarrow U$ and holomorphic on U that fixed point r , then:

- 1) r is interior fixed point for ψ if $r \in U$
- 2) r is exterior fixed point ψ if $r \notin U$

Proposition (1.12):

Then 0 is interior fixed point and $\frac{2}{\lambda}$ is exterior fixed point for ϕ_λ .

Proof :

Since ϕ_λ has two fixed points $z_1 = 0, z_2 = \frac{2}{\lambda}$, $|z_1| = |0| = 0 < 1$. Thus z_1 is interior fixed point for ϕ_λ

Since $\lambda \in U$, then $|\lambda| < 1 < 2 = |2|$, therefore $|\lambda| < |2|$, hence $|z_2| = \left| \frac{2}{\lambda} \right| > 1$. Thus z_2 is exterior fixed point for ϕ_λ

Remark(1.13) :

ϕ_λ is its own compositional inverse in fact $\phi_\lambda(\phi_\lambda(z)) = z$ for every z in U This property of ϕ_λ is called the self-inverse property.

Remark(1.14) :

$$\text{for } \lambda \in U, \text{ then } \phi'_\lambda(0) = -1, \phi'_\lambda(\lambda) = \frac{-1}{(\lambda^2 - 1)^2}.$$

Definition(1.15) : [1.15]

Let $\psi : U \rightarrow U$ and holomorphic on U . We say that ψ is a rotation round the origin if there exists

$$\sigma \in \partial U \text{ such that } \psi(z) = \sigma z \quad (z \in U)$$

Proposition (1.12):

If $\lambda = i$, then $\phi_\lambda(z)$ is a rotation around the origin

Proof:

Since $\phi_\lambda(z) = \frac{z}{\lambda z - 1}$, since $\lambda = i$, then $\phi_\lambda(z) = -z$, let $\sigma = -1$, $\sigma \in \partial U$, hence $\phi_\lambda(z) = \sigma z$. Thus

$\phi_\lambda(z)$ is a rotation around the origin.

Theorem (1.13) : [11]

Let $\psi : U \rightarrow U$ and holomorphic on U , then ψ is elliptic if and only if ψ is automorphism that has an interior fixed point.

Proposition (1.14) :

For $\lambda \in U$, ϕ_λ is elliptic

Proof :

From (1-14), ϕ_λ is automorphism, and from (1-14) ϕ_λ has an interior fixed

Point, hence ϕ_λ is elliptic.

Definition(1.15): [11]

A linear fractional transformation is a mapping of the form $\tau(z) = \frac{az + b}{cz + d}$, where a, b, c , and d are

complex numbers and $\tau(z) = \frac{az + b}{cz + d}$ we sometime denote it by $\tau_A(z)$ where A is the non-singular 2×2

complex matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Proposition (1.16) :

ϕ_λ is a linear fractional transformation.

Proof :

Since $\phi_\lambda(z) = \frac{z}{\lambda z - 1} = \frac{az + b}{cz + d}$ such that $a = 1$, $b = 0$, $c = \lambda$, $d = -1$ and a, b, c , and d are complex

numbers and $A = \begin{bmatrix} 1 & 0 \\ \lambda & -1 \end{bmatrix}$, hence by (1.15) ϕ_λ is a linear fractional transformation.

Section Two

Definition(۲.۱): [۴]

Let U denote the unit ball in the complex plane, the Hardy space H^2 is the set of functions

$f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n) z^n$ holomorphic on U such that $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ with $f^{\wedge}(n)$ denotes then the Taylor coefficient of f .

Remark (۲.۲) : [۱]

We can define an inner product of the Hardy space functions as follows:

$f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n) z^n$ and $g(z) = \sum_{n=0}^{\infty} g^{\wedge}(n) z^n$, then inner product of f and g is:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} f^{\wedge}(n) \overline{g^{\wedge}(n)}$$

Example (۲.۳) : [۱, ۱]

Let $\alpha \in U$ and $k_{\alpha}(z) = \frac{1}{1 - \overline{\alpha}z}$ ($z \in U$). Since $\alpha \in U$ then $|\alpha| < 1$, hence the geometric series

$\sum_{n=0}^{\infty} |\alpha|^{2n}$ is convergent and thus $k_{\alpha} \in H^2$ and $k_{\alpha}(z) = \overline{\alpha} z^n$.

Definition(۲.۴) : [۴]

Let $\psi : U \rightarrow U$ and holomorphic on U , the composition operator C_{ψ} induced by ψ is defined on H^2

by the equation $C_{\psi} f = f \circ \psi$ ($f \in H^2$)

Definition(۲.۵) : [۳]

Let T be a bounded operator on a Hilbert space H , then the norm

of an operator T is defined by $\|T\| = \sup\{\|Tf\| : f \in H, \|f\| = 1\}$.

Littlewood's Subordination principle (۲.۶) : [۱, ۱]

Let $\psi : U \rightarrow U$ and holomorphic on U with $\psi(0) = 0$, then for each $f \in H^2$, $f \circ \psi \in H^2$ and

$\|f \circ \psi\| \leq \|f\|$. The goal of this theorem $C_{\psi} : H^2 \rightarrow H^2$.

Definition(۳.۷) :

The composition operator C_{ϕ_λ} induced by ϕ_λ is defined on H^2 by the equation $C_{\phi_\lambda} f = f \circ \phi_\lambda$

Proposition(۳.۸) :

If $\lambda \in U$, then for each $f \in H^2$, $f \circ \phi_\lambda \in H^2$ and $\|f \circ \phi_\lambda\| \leq \|f\|$

Proof :

Since $\phi_\lambda : U \rightarrow U$ and holomorphic on U with $\phi_\lambda(0) = 0$, then by (۳-۶)

$f \in H^2$, $f \circ \phi_\lambda \in H^2$ and $\|f \circ \phi_\lambda\| \leq \|f\|$, hence $C_{\phi_\lambda} : H^2 \rightarrow H^2$

Remark (۳.۹) : [۴]

۱) One can easily show that $C_\kappa C_\psi = C_{\psi \circ \kappa}$ and hence $C_\psi^n = C_\psi C_\psi \cdots C_\psi$

$$= C_{\psi \circ \psi \circ \cdots \circ \psi} = C_{\psi_n}$$

۲) C_ψ is the identity operator on H^2 if and only if ψ is identity map from U into U and holomorphic on U .

۳) It is simple to prove that $C_\kappa = C_\psi$ if and only if $\kappa = \psi$.

Theorem (۳.۱۰) : [۱۱]

Let $\psi : U \rightarrow U$ and holomorphic on U . C_ψ is an invertible operator on H^2 if and only if ψ

automorphism of U and $C_\psi^{-1} = C_{\psi^{-1}}$

Proposition(۳.۱۱) :

If $\lambda \in U$, then C_{ϕ_λ} is an invertible operator on H^2

Proof :

Since ϕ_λ is automorphism of U by (۱-۴) and $C_{\phi_\lambda}^{-1} = C_{\phi_\lambda^{-1}} = C_{\phi_\lambda}$ by (۱-۹), hence C_{ϕ_λ} is an invertible operator on H^2 .

Definition(۳.۱۲): [۱۲]

Let T be an operator on a Hilbert space H , The operator T^* is the adjoint of T if $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for each $x, y \in H$.

Theorem (3.13) : [3]

$V_{\alpha \in U} \{K_\alpha\}$ forms a dense subset of H^2 .

Theorem (3.14) : [4]

Let $\psi : U \rightarrow U$ and holomorphic on U , then for all $\alpha \in U$

$$C_\psi^* K_\alpha = K_{\psi(\alpha)}$$

Definition(3.15): [5]

Let $g \in H^\infty$, the Toeplitz operator T_g is the operator on H^2 given by :

$$(T_g f)(z) = g(z) f(z) \quad (f \in H^2, z \in U)$$

Remark (3.16) : [6]

For each $f \in H^2$, it is well- know that $T_h^* f = T_h^- f$, such that $h \in H^\infty$.

Proposition(3.17) :

Let $C_{\phi_\lambda}^* = T_g C_\gamma T_h$, where $h(z) = (\lambda z - 1)$, $g(z) = -1$, $\gamma(z) = \bar{\lambda} - z$

Proof :

By (3.16), $T_h^* f = T_h^- f$ for each $f \in H^2$. Hence for all $\alpha \in U$,

$$\langle T_h^* f, k_\alpha \rangle = \langle T_h^- f, k_\alpha \rangle = \langle f, T_h^* k_\alpha \rangle \dots \dots (2-1)$$

On the other hand ,

$$\langle T_h^* f, k_\alpha \rangle = \langle f, T_h f \rangle = \langle f, h(\alpha) k_\alpha \rangle \dots \dots (2-2)$$

From (3.1) and (3.2) one can see that $T_h^* k_\alpha = h(\alpha) k_\alpha$. Hence $T_h^* k_\alpha = \overline{h(\alpha)} k_\alpha$.

Calculation give

$$\begin{aligned} C_{\phi_\lambda}^* k_\alpha(z) &= k_{\phi_\lambda(\alpha)}(z) \\ &= \frac{1}{1 - \overline{\phi_\lambda(\alpha)} z} = \frac{1}{1 - \frac{\overline{\alpha z}}{\lambda \alpha - 1}} \\ &= \frac{1}{\frac{\lambda \alpha - 1 - \alpha z}{\lambda \alpha - 1}} = \frac{\overline{\lambda \alpha - 1}}{\lambda \alpha - 1 - \alpha z} = \frac{\overline{(\lambda \alpha - 1)}}{-\left(1 - \overline{\alpha(\lambda - z)}\right)} \end{aligned}$$

$$\begin{aligned}
&= \overline{(\lambda\alpha - 1)} \cdot (-1) \cdot \frac{1}{1 - \overline{\alpha}(\overline{\lambda} - z)} \\
&= \overline{h(\alpha)} \cdot T_g k_\alpha(\gamma(z)) = T_g \overline{h(\alpha)} k_\alpha(\gamma(z)) \\
&= T_g \overline{h(\alpha)} C_\gamma k_\alpha(z) = T_g C_\gamma \overline{h(\alpha)} k_\alpha(z) \\
&= T_g C_\gamma T_h^* k_\alpha(z), \text{ therefore}
\end{aligned}$$

$$C_{\phi_\lambda}^* k_\alpha(z) = T_g C_\gamma T_h^* k_\alpha(z) \quad (z \in U).$$

$$\text{But } \overline{V_{\alpha \in U} \{K_\alpha\}} = H^2, \text{ then } C_{\phi_\lambda}^* = T_g C_\gamma T_h^*$$

Definition (3.1.11) : [11]

Let T be an operator on a Hilbert space H , T is called normal operator if $T T^* = T^* T$

Theorem (3.1.12) : [12]

Let $\psi : U \rightarrow U$ and holomorphic on U , then C_ψ is normal if and only if $\psi(z) = \beta z$ for some β , $|\beta| = 1$

Proposition (3.2.1) :

If $\lambda = 0$, then C_{ϕ_λ} is normal composition operator.

Proof :

Since $\phi_\lambda(z) = \frac{z}{\lambda z - 1}$, since $\lambda = 0$, then $\phi_\lambda(z) = -z = \beta z$ such that $\beta = -1, |\beta| = 1$.

Hence by (3.1.12) C_{ϕ_λ} is normal composition operator.

Definition (3.2.2) : [12]

Let T be an operator on a Hilbert space H , T is called an algebraic operator on H if there exists non-zero complex polynomial P such that $P(T) = 0$, and T is called idempotent on H if $T^2 = T$.

Proposition (3.2.3) :

Let $\lambda \in U$, then C_{ϕ_λ} is not idempotent composition operator on H^2

Proof :

Since $C_{\phi_\lambda}^2 = C_{\phi_\lambda} C_{\phi_\lambda} = C_{\phi_\lambda \circ \phi_\lambda}$ by (2.9), since $\phi_\lambda \circ \phi_\lambda = I$ by (self-inverse property). Hence $C_{\phi_\lambda \circ \phi_\lambda} = C_I$ and $C_I = I$ by (2.9), hence $C_{\phi_\lambda}^2 = I \neq C_{\phi_\lambda}$. Hence C_{ϕ_λ} is not idempotent composition operator on H^2

Proposition(2.23) :

Let $\lambda \in U$, then C_{ϕ_λ} is an algebraic composition operator.

Proof :

Take

$P(z) = z^4 - z^2$, note that for all $f \in H^2, z \in U$

$$\begin{aligned} P(C_{\phi_\lambda}) f(z) &= (C_{\phi_\lambda}^4 - C_{\phi_\lambda}^2) f(z) \\ &= C_{\phi_\lambda}^4 f(z) - C_{\phi_\lambda}^2 f(z) \end{aligned}$$

Then, it is easily seen that $C_{\phi_\lambda}^4 = C_{\phi_\lambda}^2 = I$ (by the self-inverse property).

Hence $P(C_{\phi_\lambda}) = 0$. Since P is non-zero complex polynomial, then C_{ϕ_λ} is an algebraic composition operator.

Definition (2.24) : [14]

Let $\psi: U \rightarrow U$ and holomorphic on U , the eigenvalue equation for the composition operator is define by $C_\psi f = \kappa f$ or $f \circ \psi = \kappa f$.

Theorem (2.25) : [14]

Let $\psi: U \rightarrow U$ and holomorphic on U , and that fixes the point $p \in U$ and suppose that $C_\psi f = \kappa f$ for some non-constant $f \in H^2$ and some $\kappa \in \mathbb{C}$. Then $\kappa = (\psi'(p))^n$ for some $n = 1, 2, 3, \dots$

Proposition(2.26) :

If $\lambda \in U$, then $(-1)^n$ is an eigenvalue of C_{ϕ_λ} for some $n = 1, 2, 3, \dots$

Proof :

Since $\phi_\lambda(z) = \frac{z}{\lambda z - 1}$, $\phi'_\lambda(z) = \frac{(\lambda z - 1)(1) - (\lambda z)}{(\lambda z - 1)^2} = \frac{-1}{(\lambda z - 1)^2}$, and since ϕ_λ fixed the point $1 \in U$,

and by (2.25) $\kappa = (\phi'_\lambda(1))^n = (-1)^n$ is an eigenvalue of C_{ϕ_λ} for some $n = 1, 2, 3, \dots$

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المؤثر التركيبي C_{ϕ_λ} المتولد بالدالة ϕ_λ

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المستخلص

ليكن U يرمز إلى كرة الوحدة في المستوى العقدي، إن فضاء هاردي H^2 هو مجموعة كل الدوال $f(z) = \sum_{n=0}^{\infty} f^{\wedge}(n) z^n$

التحليلية على U بحيث أن $\sum_{n=0}^{\infty} |f^{\wedge}(n)|^2 < \infty$ ، يرمز إلى معاملات تيلر النونية.

لتكن $\psi: U \rightarrow U$ دالة تحليلية على U ، المؤثر التركيبي المحتث من ψ يعرف على فضاء هاردي H^2 بواسطة:

$$C_\psi f = f \circ \psi \quad (f \in H^2).$$

درسنا في هذا البحث المؤثر التركيبي المحتث من الدالة ϕ_λ حيث ناقشنا المؤثر المرافق للمؤثر التركيبي المحتث من الدالة ϕ_λ .

بالإضافة إلى ذلك نظرنا إلى بعض النتائج المعروفة وحاولنا الحصول على نتائج مناظرة لنتمكن من ملاحظة كيفية تغير النتائج عندما تتغير

الدالة التحليلية ψ .

ومن أجل جعل مهمة القارئ أكثر سهولة، عرضنا بعض النتائج المعروفة عن المؤثرات التركيبية وعرضنا براهين مفصلة وكذلك

برهنا بعض النتائج.